TRANSFERS ON MILNOR-WITT K-THEORY

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Abstract

We study the existence of transfers on a generalization of Milnor K-theory called Milnor-Witt K-theory. We give a new proof of the fact that Milnor-Witt K-theory has geometric transfers. Moreover, we explain how our proof yields a simplification of Morel's conjecture about Bass-Tate-Kato transfers on contracted homotopy sheaves in the context of motivic homotopy theory.

Contents

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1 Introduction

1.1 Current work

In the beginning of the century, Morel (in joint work with Hopkins) defined for a field E the Milnor-Witt K-theory $\mathbf{K}_*^{\mathrm{MW}}(E)$ (see [Mor12, Definition 3.1]). This $\mathbb{Z}$-graded abelian group behaves in positive degrees like Milnor K-theory groups $\mathbf{K}_n^M(E)$ (or rather its fibre product with powers of the fundamental ideal), and in non-positive degrees like Grothendieck-Witt and Witt groups of quadratic forms, $\mathrm{GW}(E)$ and $\mathrm{W}(E)$. The Milnor-Witt K-theory was originally used for solving some splitting problems for projective modules (see e.g. the work of Barge and Morel [BM00]). Since then, Milnor-Witt K-groups have proven to be relevant for motivic homotopy and its applications in algebraic geometry.

The word "transfer" has many incarnations in mathematics. Philosophically, a transfer is a way to pass on information from one world to another. In K-theory and algebraic geometry, transfers are maps related to pushforwards or maps that go in the wrong way. For instance, in [BT73], Bass and Tate defined a map

$$\mathrm{Tr}_{x/F} : \mathbf{K}_*^{\mathrm{M}}(F(x)) \to \mathbf{K}_*^{\mathrm{M}}(F)$$

for any monogenic extension of fields $F(x)/F$. Unfortunately, the natural definition given by Bass and Tate had one issue: the map $\mathrm{Tr}_{x/F}$ may depend on the choice of generator x. This raises the question of functoriality of such transfer maps. In 1973, Bass and Tate conjectured that such transfers are well-defined but a proof appeared only a decade later in the work of Kato [Kat80].

The study of transfers has a long history in motivic homotopy theory (see [FSV00, Dé12, Fas08, GP18, BCD$^+$20, Fel21]). In [Mor12, Chapter 4], Morel introduced transfers on the Milnor-Witt K-theory of a field. Following ideas of Bass and Tate [BT73], one can define geometric transfer maps

$$\begin{gathered}\mathrm{Tr}_{x_1,\dots,x_r/E} = \mathrm{Tr}_{x_r/E(x_1,\dots,x_{r-1})} \circ \cdots \circ \mathrm{Tr}_{x_1/E} : \\ \mathbf{K}_*^{\mathrm{MW}}(E(x_1,\dots,x_r), \omega_{E(x_1,\dots,x_r)/E}) \to \mathbf{K}_*^{\mathrm{MW}}(E)\end{gathered}$$

on $\mathbf{K}_*^{\mathrm{MW}}$ for finite extensions $E(x_1,\dots,x_r)/E$ (see the next section for more details). Morel proved in [Mor12, Chapter 4] that such transfers are well-defined and functorial. The relevance of $\omega_{E(x_1,\dots,x_r)/E}$ for making the transfers independent of choices of generating elements is hinted by the fact that the naive definition of the residue map ∂_v^π of a discrete valuation depends on the choice of prime π (see [Mor12, Remark 3.20]).

In this article, we give a new (shorter) proof of this result:

Theorem 1 (Theorem 2.17)**.** *Let $E(x_1,\dots,x_r)/E$ be a finite extension of fields. The transfer map*

$$\mathrm{Tr}_{x_1,\dots,x_r/E} : \mathbf{K}_*^{MW}(E(x_1,\dots,x_r),\omega_{E(x_1,\dots,x_r)/E}) \to \mathbf{K}_*^{MW}(E)$$

does not depend on the choice of the generating system $(x_1,\dots,x_r)$.

The idea is to reduce to the case of p-primary fields (see Definition 2.3) then study the transfers manually, as Kato originally did for Milnor K-theory (see [GS17] for an elementary exposition).

Moreover, this proof applies to the study of a conjecture of Morel about the existence of transfer maps for (contracted) homotopy sheaves:

Theorem 2 (Theorem 3.26)**.** *In order to prove that a contracted homotopy sheaf* M_{-1} *has functorial transfers, it suffices to consider the case of* p*-primary fields (where* p *is a prime number).*

1.2 Outline of the paper

In Subsection 2.1, we recall some properties of fields called p-primary fields. For p a prime number, a p-primary field has no nontrivial finite extension prime to p (see Definition 2.3). In Subsection 2.2 and Subsection 2.3, we give the basic definitions of Milnor-Witt K-theory In Subsection 3.1 and Subsection 3.2, we prove that Milnor-Witt K-theory has transfer maps which are functorial. The proof is similar to the original proof of Kato for Milnor K-theory: we reduce to the case of p-primary fields then study the transfers manually. In Subsection 3.3, we end with a discussion of a conjecture of Morel in motivic homotopy theory by applying ideas from Subsection 3.1.

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2 Definitions

Notation

Throughout the paper, we fix a (commutative) field k and we assume moreover that k is perfect (of arbitrary characteristic).

By a field E over k, we mean *a finitely generated extension of fields* E/k.

Let E be a field (over k) and v a valuation on E. We will always assume that v is discrete. We denote by $\mathcal{O}_v$ its valuation ring, by $\mathfrak{m}_v$ its maximal ideal and by $\kappa(v)$ its residue class field. We consider only valuations of geometric type, that is we assume: $k \subset \mathcal{O}_v$, the residue field $\kappa(v)$ is finitely generated over k and satisfies $\operatorname{tr.deg}_k(\kappa(v))+1 = \operatorname{tr.deg}_k(E)$.

Let $f : X \to Y$ be a morphism of schemes. Denote by $\mathcal{L}_f$ (or $\mathcal{L}_{X/Y}$) the virtual vector bundle over Y associated with the cotangent complex of f, and by ω_f (or $\omega_{X/Y}$) its determinant. Recall that if $p : X \to Y$ is a smooth morphism, then $\mathcal{L}_p$ is (isomorphic to) $\mathcal{T}_p = \Omega_{X/Y}$ the space of relative (Kähler) differentials. If $i : Z \to X$ is a regular closed immersion, then $\mathcal{L}_i$ is the normal cone $-\mathcal{N}_Z X$. If f is the composite $Y \xrightarrow{i} \mathbb{P}^n_X \xrightarrow{p} X$ with p and i as previously (in other words, if f is lci quasi-projective), then $\mathcal{L}_f$ is isomorphic to the virtual tangent bundle $i^*\mathcal{T}_{\mathbb{P}^n_X/X} - \mathcal{N}_Y(\mathbb{P}^n_X)$. In practice, we mostly work with smooth schemes hence every map (between smooth schemes) is lci quasi-projective.

Let X be a scheme and $x \in X$ a point. Specializing the previous notations, we denote by $\mathcal{L}_x = \mathcal{L}_{\operatorname{Spec}(\kappa(x))/X} = (\mathfrak{m}_x/\mathfrak{m}_x^2)^\vee$ and ω_x its determinant. Similarly, let v a discrete valuation on a field, we denote by ω_v the line bundle $(\mathfrak{m}_v/\mathfrak{m}_v^2)^\vee$.

Let E be a field. We denote by $\mathrm{GW}(E)$ the Grothendieck-Witt ring of symmetric bilinear forms on E (another equivalent definition of $\mathrm{GW}(E)$ is given in [Mor12, Lemma 3.9]. This is well-defined in characteristic 2 according to the work of Morel). For any $a \in E^*$, we denote by $\langle a \rangle$ the class of the symmetric bilinear form on E defined by $(X,Y) \mapsto aXY$ and, for any natural number n, we put $n_\epsilon = \sum_{i=1}^n \langle -1 \rangle^{i-1}$. Recall that if n and m are two natural numbers, then $(nm)_\epsilon = n_\epsilon m_\epsilon$.

2.1 On p-primary fields

We recall some facts about fields (see [Sha82, §1] and [BT73, Section 5]). Let E be a field and p a prime number. Fix a separable closure E_s of E and consider the set of all sub-extensions of E_s that contain E and that can be realized as a union of finite prime-to-p extensions of E. Zorn's lemma implies that this set contains a maximal element $E_{\langle p \rangle}$ for the inclusion.

Proposition 2.1. *If F is a finite extension of E contained in $E_{\langle p \rangle}$, then its degree $[F : E]$ is prime to p.*

Proof. Write $F = E(x_1, \dots, x_r)$ with $x_i \in F$. Each x_i is contained in a prime-to-p extension of E hence has a degree prime to p. □

Proposition 2.2. *If F is a finite extension of $E_{\langle p \rangle}$, then its degree $[F : E_{\langle p \rangle}]$ is equal to p^n for some natural number n.*

Proof. Let x be any element in F and denote by P_x its irreducible polynomial over $E_{\langle p \rangle}$. We prove that its degree is a power of p. All the coefficients lie in a finite prime-to-p

extension of E. Without loss of generality, we may assume that $E_{\langle p\rangle}(x)$ is nontrivial. If the degree of x over $E_{\langle p\rangle}$ is prime to p, then $E_{\langle p\rangle}(x)$, which is a nontrivial extension of $E_{\langle p\rangle}$, contradicts the maximality of $E_{\langle p\rangle}$. Write $p^n m$ the degree of x over $E_{\langle p\rangle}$ with $n, m \geq 1$ and $(m,p) = 1$. Let F_N be the normal closure of F in E_s ; it is a Galois extension of $E_{\langle p\rangle}$ whose degree over $E_{\langle p\rangle}$ is divisible by $p^n m$. If $m \neq 1$, then a Sylow p-subgroup $S(p)$ of $\mathrm{Gal}(F_N/E_{\langle p\rangle})$ is a nontrivial proper subgroup and the fixed field $F_N^{S(p)}$ is a nontrivial prime-to-p extension of $E_{\langle p\rangle}$, which is absurd. Thus $m = 1$ and the result follows. □

The previous result leads to the following definition.

DEFINITION 2.3. A field that has no nontrivial finite extensions of degree prime to a prime number p is called p-primary.

PROPOSITION 2.4. *Let F be a nontrivial finite extension of $E_{\langle p\rangle}$ contained in E_s and let p^n be the degree $[F : E_{\langle p\rangle}]$. Then there is a tower of fields*

$$E_{\langle p\rangle} = F_1 \subset F_2 \subset \cdots \subset F_n = F$$

such that $[F_i : F_{i-1}] = p$.

Proof. We prove the result by induction on n. We need to find a subfield K of F whose degree over $E_{\langle p\rangle}$ is p^{n-1}. The group $G = \mathrm{Gal}(E_s/E_{\langle p\rangle})$ is a pro-p-group since all finite extensions of $E_{\langle p\rangle}$ contained in E_s are p-power extensions. Galois theory implies that F is the fixed subfield of a subgroup H of G with $[G : H] = p^n$. We will find a subgroup H_1, such that $H \subset H_1 \subset G$ and $[G : H_1] = p^{n-1}$. Letting $K = E_s^{H_1}$, we will get the desired subfield K.

The group H is subgroup of G of finite index hence is open. By the class equation, it also follows that H has only a finite number of conjugates in G. Let $H' = \cap_{x\in G} x^{-1}Hx$, then H' is an open normal subgroup of G containing H. The group G/H' is a finite p-group containing H/H'. By the Sylow theorems, we can find H_1, normal in G, with $H \subset H_1 \subset G$ and $[G : H_1] = p^{n-1}$. This ends to proof. □

Similarly, we obtain the following result.

LEMMA 2.5. *Let p be a prime number and E a p-primary field. Let F/E be a finite extension (not necessarily separable).*

1. *The field F inherits the property of having no nontrivial finite extension of degree prime to p.*

2. *If $F \neq E$, then there exists a subfield $E \subset F' \subset F$ such that F'/E is a normal extension of degree p.*

Proof. (See also [GS17, Lemma 7.3.7])

1. Let L/F be a finite extension of degree prime to p.

   If L/F is separable, take the Galois closure $\tilde{L}$. Since E is a p-primary field, the fixed field of a p-Sylow subgroup in $\mathrm{Gal}(\tilde{L}/E)$ must equal E, hence $L = F$.

   If F/E is purely inseparable, then L/F must be separable, thus L/E has a subfield $L_0 = E$ separable unless $L = F$.

   If F/E is separable but L/F is not, then we may assume that L/F is purely inseparable. Taking a normal closure $\tilde{L}$, the fixed field of $\mathrm{Aut}_E(\tilde{L})$ defines a nontrivial prime to p extension of E unless $L = F$.

2. The second statement is straightforward in the case when the extension F/E is purely inseparable (see [Sta21, Section 9.14, tag 09HD]), so by replacing F with the maximal separable subextension of F/E, we may assume that F/E is a separable extension. Denote by $\tilde{F}$ the Galois closure of F. The first statement implies that the Galois group $G := \mathrm{Gal}(\tilde{F}/E)$ is a p-group. Let H be a maximal subgroup of G containing $\mathrm{Gal}(\tilde{F}/F)$. By the theory of finite p-groups (see [Suz82, Corollary of Theorem 1.6]), it is a normal subgroup of index p in G, so we may take F' to be its fixed field.

□

2.2 Milnor-Witt K-theory

We describe the Milnor-Witt K-theory, as defined by Morel (see [Mor12, §3] or [Fas20, §1.1] or [Fel20a, §1]).

Definition 2.6. Let E be a field. The Milnor-Witt K-theory algebra of E is defined to be the quotient of the free $\mathbb{Z}$-graded algebra generated by the symbols $[a]$ of degree 1 for any $a \in E^\times$ and a symbol $\boldsymbol{\eta}$ in degree -1 by the following relations:

- $[a][1-a] = 0$ for any $a \in E^\times \setminus \{1\}$.
- $[ab] = [a] + [b] + \boldsymbol{\eta}[a][b]$ for any $a, b \in E^\times$.
- $\boldsymbol{\eta}[a] = [a]\boldsymbol{\eta}$ for any $a \in E^\times$.
- $\boldsymbol{\eta}(\boldsymbol{\eta}[-1] + 2) = 0$.

The relations being homogeneous, the resulting algebra is $\mathbb{Z}$-graded. We denote it by $\mathbf{K}_*^{\mathrm{MW}}(E)$.

REMARK 2.7.
- By definition of the Milnor K-theory of a field $\mathbf{K}_n^{\mathrm{M}}(E)$, we have a natural isomorphism

$$\mathbf{K}_n^{\mathrm{MW}}(E)/\boldsymbol{\eta} \simeq \mathbf{K}_n^{\mathrm{M}}(E)$$

given by $[a] \mapsto [a], \boldsymbol{\eta} \mapsto 0$ (for any natural number $n \geq 0$).

- If $\phi : E \to F$ is a field extension, then we have a map

$$\mathrm{res}_{F/E} : \mathbf{K}_*^{\mathrm{MW}}(E) \to \mathbf{K}_*^{\mathrm{MW}}(F)$$

given by $[a] \mapsto [\phi(a)], \boldsymbol{\eta} \mapsto \boldsymbol{\eta}$, and called the restriction map.

2.8. NOTATION We will use the following notations.

- $[a_1, \dots, a_n] = [a_1] \dots [a_n]$ for any $a_1, \dots, a_n \in E^\times$.
- $\langle a \rangle = 1 + \boldsymbol{\eta}[a]$ for any $a \in E^\times$.
- $\epsilon = -\langle -1 \rangle$.
- $n_\epsilon = \sum_{i=1}^{n} \langle (-1)^{i-1} \rangle$ for any $n \geq 0$, and $n_\epsilon = \epsilon(-n)_\epsilon$ if $n < 0$.

EXAMPLE 2.9. If $a \in E^\times$, we also denote by $\langle a \rangle$ the class of the bilinear form $(X, Y) \mapsto aXY$ in $\mathrm{GW}(E)$, the Grothendieck-Witt group of E [Lam05, Chapter 1]. According to [Mor12, Lemma 3.10], the map $\langle a \rangle \mapsto 1 + \boldsymbol{\eta}[a]$ defines a canonical isomorphism

$$\mathrm{GW}(E) \simeq \mathbf{K}_0^{\mathrm{MW}}(E)$$

and the multiplication by $\boldsymbol{\eta}$ induces an isomorphism

$$W(E) \simeq \mathbf{K}_n^{\mathrm{MW}}(E)$$

for any $n < 0$ (where $W(E)$ is the Witt group of E, see [Lam05, Chapter 1]).

2.10. TWISTED MILNOR-WITT K-THEORY Let E be a field and $\mathcal{L}_E$ a 1-dimensional vector space over E. The group $E^\times$ of invertible elements of E acts naturally on $\mathcal{L}_E^\times$, the set of non-zero elements in $\mathcal{L}_E$; hence the free abelian group $\mathbb{Z}[\mathcal{L}_E^\times]$ is a $\mathbb{Z}[E^\times]$-module. Define

$$\mathbf{K}_n^{\mathrm{MW}}(E, \mathcal{L}_E) = \mathbf{K}_n^{\mathrm{MW}}(E) \otimes_{\mathbb{Z}[E^\times]} \mathbb{Z}[\mathcal{L}_E^\times].$$

Let $\mathcal{L}_E$ and $\mathcal{L}'_E$ be two line bundles over E, and n, n' two integers. The product of the Milnor-Witt K-theory groups induces a product

$$\begin{gathered}\mathbf{K}_n^{\mathrm{MW}}(E,\mathcal{L}_E)\otimes\mathbf{K}_{n'}^{\mathrm{MW}}(E,\mathcal{L}'_E)\to\mathbf{K}_{n+n'}^{\mathrm{MW}}(E,\mathcal{L}_E\otimes\mathcal{L}'_E)\\ (x\otimes l,x'\otimes l')\mapsto(xx')\otimes(l\otimes l').\end{gathered}$$

2.11. Residue morphisms (see [Mor12, Theorem 3.15]) Let E be a field endowed with a discrete valuation v. We choose a uniformizing parameter π. As in the classical Milnor K-theory, we can define a residue morphism

$$\partial_v^\pi:\mathbf{K}_*^{\mathrm{MW}}(E)\to\mathbf{K}_{*-1}^{\mathrm{MW}}(\kappa(v))$$

commuting with the multiplication by $\boldsymbol{\eta}$ and satisfying the following two properties:

- $\partial_v^\pi([\pi,a_1,\dots,a_n])=[\overline{a_1},\dots,\overline{a_n}]$ for any $a_1,\dots,a_n\in\mathcal{O}_v^\times$.
- $\partial_v^\pi([a_1,\dots,a_n])=0$ for any $a_1,\dots,a_n\in\mathcal{O}_v^\times$.

The main difference between Milnor and Milnor-Witt K-theory is that this morphism does depend on the choice of π. Indeed, if we consider another uniformizer π' and write $\pi'=u\pi$ where u is a unit, then we have $\partial_v^\pi(x)=\langle u\rangle\partial_v^{\pi'}(x)$ for any $x\in\mathbf{K}_*^{\mathrm{MW}}(E)$. Nevertheless, by twisting by the dual of the normal cone $\omega_v=(\mathfrak{m}_v/\mathfrak{m}_v^2)^\vee$, we can define a twisted residue morphism that does not depend on π:

$$\begin{gathered}\partial_v:\mathbf{K}_*^{\mathrm{MW}}(E,\mathcal{L}_E)\to\mathbf{K}_{*-1}^{\mathrm{MW}}(\kappa(v),\omega_v\otimes\mathcal{L}_{\kappa(v)})\\ x\otimes l\mapsto\partial_v^\pi(x)\otimes(\bar{\pi}^*\otimes l)\end{gathered}$$

where $\mathcal{L}_E$ and $\mathcal{L}_{\kappa(v)}$ are the pullbacks of a free rank 1 module $\mathcal{L}$ over $\mathcal{O}_v$, $\bar{\pi}$ is the canonical projection of π modulo $\mathfrak{m}_v$ and $\bar{\pi}^*$ the dual of $\bar{\pi}$ (i.e. its canonical associated linear form).

2.3 Transfers

Recall the definition of transfers on Milnor-Witt K-theory; the definition for Milnor-Witt K-theory is analogous to the definition for Milnor K-theory of Bass and Tate (see [BT73], see also [GS17]).

Theorem 2.12 (Homotopy invariance). *Let F be a field and $F(t)$ the field of rational functions with coefficients in F in one variable t. We have a split short exact sequence*

$$0\to\mathbf{K}_*^{MW}(F)\overset{\mathrm{res}}{\to}\mathbf{K}_*^{MW}(F(t))\overset{d}{\to}\bigoplus_{x\in(\mathbb{A}_F^1)^{(1)}}\mathbf{K}_{*-1}^{MW}(\kappa(x),\omega_x)\to 0$$

where $\mathrm{res}=\mathrm{res}_{F(t)/F}$ *is the restriction map defined in 2.7 and* $d=\bigoplus_{x\in(\mathbb{A}_F^1)^{(1)}}\partial_x$ *is the sum of the residue maps defined in 2.11.*

Proof. See [Mor12, Theorem 3.24] (actually, Morel does not use twisted sheaves but chooses a generator for each ω_x instead, which is equivalent. Note also that the choice of a generator for each ω_x is the same as a choice of uniformizer for the valuations corresponding to the closed points). □

2.13. Let $\phi : E \to F$ be a monogenic finite field extension and choose $x \in F$ such that $F = E(x)$. The homotopy exact sequence implies that for any $\beta \in \mathbf{K}_*^{\mathrm{MW}}(F, \omega_{F/k})$ there exists $\gamma \in \mathbf{K}_*^{\mathrm{MW}}(E(t), \omega_{E(t)/k})$ with the property that $d(\gamma) = \beta$ (note that we identify the element β with a tuple in a direct sum of Milnor-Witt groups which has one entry β and all other entries 0). Now the valuation at ∞ yields a morphism

$$\partial_\infty : \mathbf{K}_{*+1}^{\mathrm{MW}}(E(t), \omega_{E(t)/k}) \to \mathbf{K}_*^{\mathrm{MW}}(E, \omega_{E/k})$$

which vanishes on the image of $\mathrm{res}_{E(t)/E}$. We denote by $\mathrm{Tr}_{x/E}(\beta)$ the element $-\partial_\infty(\gamma)$; it does not depend on the choice of γ. This defines a group morphism

$$\mathrm{Tr}_{x/E} : \mathbf{K}_*^{\mathrm{MW}}(E(x), \omega_{F/k}) \to \mathbf{K}_*^{\mathrm{MW}}(E, \omega_{E/k})$$

called the *transfer map* and also denoted by $\mathrm{Tr}_{x/E}$. The following result completely characterizes the transfer maps.

LEMMA 2.14 (projection formula). *Keeping the previous notations, let $\alpha \in \mathbf{K}_*^{MW}(E)$ and $\beta \in \mathbf{K}_*^{MW}(E(x))$. We then have*

$$\mathrm{Tr}_{x/E}(\mathrm{res}_{E(x)/E}(\alpha) \cdot \beta) = \alpha \cdot \mathrm{Tr}_{x/E}(\beta).$$

Proof. It suffices to prove the result for $\alpha = [u]$ with $u \in E^\times$. Let $\gamma \in \mathbf{K}_*^{\mathrm{MW}}(E(t), \omega_{E(t)/k})$ such that $d(\gamma) = \beta$. It follows from [Mor12, Proposition 3.17] that for any valuation v, we have $\partial_v([u]\gamma) = -\langle -1\rangle[\overline{u}]\partial_v(\gamma)$. Thus $-\langle -1\rangle[u]\gamma$ is a lift of $[u]\beta$ and $\partial_\infty(-\langle -1\rangle[u]\gamma) = [u]\partial_v(\gamma)$. Thus $\mathrm{Tr}_{x/E}(\mathrm{res}_{E(x)/E}(\alpha) \cdot \beta) = \alpha \cdot \mathrm{Tr}_{x/E}(\beta)$. □

LEMMA 2.15. *Keeping the previous notations, let*

$$d = (\bigoplus_x d_x) \oplus d_\infty : \mathbf{K}_{*+1}^{MW}(E(t), \omega_{F(t)/k}) \to (\bigoplus_x \mathbf{K}_*^{MW}(E(x), \omega_{E(x)/k})) \oplus \mathbf{K}_*^{MW}(E, \omega_{E/k})$$

be the total twisted residue morphism (where x runs through the set of monic irreducible polynomials in $E(t)$). Then, the transfer maps $\mathrm{Tr}_{x/E}$ are the unique morphisms such that $\sum_x (\mathrm{Tr}_{x/E} \circ d_x) + d_\infty = 0$.

Proof. Straightforward (see [Mor12, §4.2]). □

DEFINITION 2.16. Let $F = E(x_1, x_2, \dots, x_r)$ be a finite extension of a field E and consider the chain of subfields

$$E \subset E(x_1) \subset E(x_1, x_2) \subset \cdots \subset E(x_1, \dots, x_r) = F.$$

Define by induction:

$$\mathrm{Tr}_{x_1,\dots,x_r/E} = \mathrm{Tr}_{x_r/E(x_1,\dots,x_{r-1})} \circ \cdots \circ \mathrm{Tr}_{x_2/E(x_1)} \circ \mathrm{Tr}_{x_1/E}$$

We give an elementary proof of the fact that the definition does not depend on the choice of the factorization (see [Mor12, Theorem 4.27] for the original proof):

THEOREM 2.17. *Let $F = E(x_1, \ldots, x_r)/E$ be a finite field extension. Then the map*

$$\mathrm{Tr}_{x_1,\ldots,x_r/E} : \mathbf{K}^{MW}_*(F) \to \mathbf{K}^{MW}_*(E)$$

does not depend on the choice of the generating system $(x_1, \ldots, x_r)$.

3 Proof of the main theorem

3.1 Reduction to the p-primary case

We begin with a series of lemmas aimed at reducing Theorem 2.17 to the case of p-primary fields.

LEMMA 3.1. *Let F/E be a finite extension of degree n of characteristic zero fields and consider the transfer map $\mathrm{Tr}_{F/E} : \mathrm{GW}(F) \to \mathrm{GW}(E)$. If n is odd, then*

$$\mathrm{Tr}_{F/E}(1) = n_\epsilon.$$

If n is even, then there exist $a_1, \ldots, a_n \in E^\times$ such that

$$\mathrm{Tr}_{F/E}(1) = \textstyle\sum_i \langle a_i \rangle.$$

Proof. See [Lam05, VII.2.2]. Note that the case n even is not really much information, it is merely the diagonalizability of quadratic forms. □

LEMMA 3.2. *Let E be a field of characteristic $p > 0$. Let $\alpha \in \mathrm{GW}(E)$ be an element in the kernel of the rank morphism $\mathrm{GW}(E) \to \mathbb{Z}$. Then α is nilpotent in $\mathrm{GW}(E)$.*

Proof. The result is not surprising: in Witt rings, torsion elements are nilpotent, and in characteristic $p > 0$, then kernel of the rank morphism is torsion.

We give a detailed proof following [LYZ19, Lemma B.4]. As the set of nilpotent elements in the commutative ring $\mathrm{GW}(E)$ is an ideal, we may assume $\alpha = \langle t \rangle - 1$ where $t \in E^\times$. We have $(1 + \alpha)^2 = \langle t^2 \rangle = 1$, so that $\alpha^2 = -2\alpha$. By induction, we get $\alpha^n = (-2)^{n-1}\alpha$ for $n \geq 1$: we have to show that α is annihilated by a power of two. If $p = 2$, $2\alpha = 0$ holds (see [Mor12, Lemma 3.9]), i.e. $\alpha^2 = 0$. Now we assume $p \geq 3$ so that there is no danger thinking in terms of usual quadratic forms. We first consider $\mu := \langle -1 \rangle - 1 \in \mathrm{GW}(\mathbb{F}_p)$. The quadratic form $-x^2 - y^2$ over $\mathbb{F}_p$ represents 1 (see [Ser77, Proposition 4,§IV.1.7]) so that $\langle -1 \rangle + \langle -1 \rangle = \langle 1 \rangle + \langle 1 \rangle \in \mathrm{GW}(\mathbb{F}_p)$, i.e. $2\mu = 0 \in \mathrm{GW}(\mathbb{F}_p)$,which gives $\mu^2 = 0$. Let $t \in E^\times$ be any nonzero element in an extension

E of $\mathbb{F}_p$. The quadratic form $q(x,y) := x^2 - y^2 = (x+y)(x-y)$ represents t (this is $q((1+t)/2,(1-t)/2))$, which easily implies that $\langle 1\rangle + \langle -1\rangle = \langle t\rangle + \langle -t\rangle$ (see also [Mor12, Lemma 3.7]. This is equivalent to saying $(2+\mu)\alpha = 0 \in \mathrm{GW}(E)$. It follows that $4\alpha = (2-\mu)(2+\mu)\alpha = 0$, and then $\alpha^3 = 0$. □

LEMMA 3.3. *Consider two finite extensions F/E and L/E of coprime degrees n and m, respectively. Let $x \in \mathbf{K}_*^{MW}(E)$ such that $\mathrm{res}_{F/E}(x) = 0 = \mathrm{res}_{L/E}(x)$. Then $x = 0$.*

Proof. Applying the transfer map to $\mathrm{res}_{F/E}(x)$ and $\mathrm{res}_{L/E}(x)$, we see that x is killed by $\mathrm{Tr}_{F/E}(1)$ and $\mathrm{Tr}_{L/E}(1)$, thanks to the projection formula (note that the transfers here are transfers for the Grothendieck-Witt ring, identified with $\mathbf{K}_0^{\mathrm{MW}}$).

In characteristic 0, up to swapping n and m, we may assume that n is odd, hence $\mathrm{Tr}_{F/E}(1) = n_\epsilon$ and $\mathrm{Tr}_{L/E}(1) = \sum_i \langle a_i\rangle$ for some $a_1, \dots, a_m \in E^\times$. Write $n = 2r+1$. There exist $a, b \in \mathbb{Z}$ such that $an + bm = r$ since n and m are coprime. Recall that the hyperbolic form $h = 1 + \langle -1\rangle$ satisfies $\langle a_i\rangle h = h$ for any i (see [Mor12, Lemma 3.7]). Hence $rh = (an_\epsilon + b\sum_i \langle a_i\rangle)h$ and $1 = n_\epsilon - rh = (1-ah)\,\mathrm{Tr}_{F/E}(1) - bh\,\mathrm{Tr}_{L/E}(1)$ kills x.

In characteristic $p > 0$, there exist two nilpotent α and α' in $\mathrm{GW}(E)$ such that $\mathrm{Tr}_{F/E}(1) = n + \alpha$ and $\mathrm{Tr}_{L/E}(1) = m + \alpha'$, according to Lemma 3.2. Hence for a natural number s large enough, the element x is killed by the coprime numbers n^{p^s} and m^{p^s} so that $x = 0$. □

LEMMA 3.4. *Let E be a field. Let $F_1, \dots, F_n$ be finite extensions of coprime degrees $d_1, \dots, d_n$. Let $\delta \in \mathbf{K}_*^{MW}(E)$ be an element such that $\mathrm{res}_{F_i/E}(\delta) = 0$ for any i. Then, δ is zero.*

Proof. This follows as in Lemma 3.3.

□

LEMMA 3.5. *Let F/E be a field extension and w be a valuation on F which restricts to a nontrivial valuation v on E with ramification index e. We have a commutative square*

$$\begin{tikzcd}
\mathbf{K}_*^{MW}(E) \arrow[r, "\partial_v"] \arrow[d, "\mathrm{res}_{F/E}"'] & \mathbf{K}_{*-1}^{MW}(\kappa(v), \omega_v) \arrow[d, "e_\epsilon \cdot \mathrm{res}_{\kappa(w)/\kappa(v)}"] \\
\mathbf{K}_*^{MW}(F) \arrow[r, "\partial_w"'] & \mathbf{K}_{*-1}^{MW}(\kappa(w), \omega_w)
\end{tikzcd}$$

where $e_\epsilon = \sum_{i=1}^{e} \langle -1\rangle^{i-1}$.

Proof. See [Mor12, Lemma 3.19]. □

LEMMA 3.6. *Let F/E be a field extension and $x \in (\mathbb{A}_E^1)^{(1)}$ a closed point. Then the following diagram*

$$\begin{tikzcd}
\mathbf{K}_*^{MW}(E(x),\omega_{E(x)/k}) \arrow[r, "\mathrm{Tr}_{x/E}"] \arrow[d, "\oplus_y \mathrm{res}_{F(y)/E(x)}"'] & \mathbf{K}_*^{MW}(E,\omega_{E/k}) \arrow[d, "\mathrm{res}_{F/E}"] \\
\bigoplus_{y\mapsto x} \mathbf{K}_*^{MW}(F(y),\omega_{F(y)/k}) \arrow[r, "\sum_y e_{y,\epsilon}\mathrm{Tr}_{y/F}"'] & \mathbf{K}_*^{MW}(F,\omega_{F/k})
\end{tikzcd}$$

is commutative, where the notation $y\mapsto x$ stands for the closed points of $\mathbb{A}^1_F$ lying above x, and $e_{y,\epsilon}=\sum_{i=1}^{e_y}\langle -1\rangle^{i-1}$ is the quadratic form associated to the ramification index of the valuation v_y extending v_x to $F(t)$.

Proof. According to Lemma 3.5, the following diagram

$$\begin{tikzcd}
\mathbf{K}_*^{\mathrm{MW}}(E(t)) \arrow[r, "\partial_x"] \arrow[d, "\mathrm{res}_{F(t)/E(t)}"'] & \mathbf{K}_{*-1}^{\mathrm{MW}}(E(x),\omega_x) \arrow[d, "\oplus_y e_{y,\epsilon}\mathrm{res}_{F(y)/E(x)}"] \\
\mathbf{K}_*^{\mathrm{MW}}(F(t)) \arrow[r, "\oplus_y\partial_y"'] & \bigoplus_{y\mapsto x}\mathbf{K}_{*-1}^{\mathrm{MW}}(F(y),\omega_y)
\end{tikzcd}$$

is commutative hence for all closed points in $\mathbb{P}^1_F$, we have

$$\partial_y(\mathrm{res}_{F(t)/E(t)}\circ\rho_x-(\oplus_y\rho_y)\circ(\oplus_y e_{y,\epsilon}\,\mathrm{res}_{F(y)/E(x)}))=0$$

and so the diagram

$$\begin{tikzcd}
\mathbf{K}_*^{\mathrm{MW}}(E(t)) \arrow[d, "\mathrm{res}_{F(t)/E(t)}"'] & \mathbf{K}_*^{\mathrm{MW}}(E(x),\omega_x) \arrow[l, "\rho_x"'] \arrow[d, "\oplus_y e_{y,\epsilon}\mathrm{res}_{F(y)/E(x)}"] \\
\mathbf{K}_*^{\mathrm{MW}}(F(t)) & \bigoplus_{y\mapsto x}\mathbf{K}_*^{\mathrm{MW}}(F(y),\omega_y) \arrow[l, "\oplus_y\rho_y"]
\end{tikzcd}$$

is commutative, where ρ_x is the splitting of Theorem 2.12. Then, we conclude according to the definition of the Bass-Tate transfer maps 2.13. □

Remark 3.7. The multiplicities e_y appearing in the previous lemma are equal to

$$[E(x):E]_i/[F(y):F]_i$$

where $[E(x):E]_i$ is the inseparable degree.

Theorem 3.8 (Strong R1c). *Let E be a field, F/E a finite field extension and L/E an arbitrary field extension. Write $F=E(x_1,\dots,x_r)$ with $x_i\in F$, $R=F\otimes_E L$ and $\psi_p : R\to R/p$ the natural projection defined for any $p\in\mathrm{Spec}(R)$. Then the diagram*

$$\begin{tikzcd}
\mathbf{K}_*^{MW}(F,\omega_{F/k}) \arrow[r, "\mathrm{Tr}_{x_1,\dots,x_r/E}"] \arrow[d, "\oplus_p\mathrm{res}_{(R/p)/F}"'] & \mathbf{K}_*^{MW}(E,\omega_{E/k}) \arrow[d, "\mathrm{res}_{L/E}"] \\
\bigoplus_{p\in\mathrm{Spec}(R)}\mathbf{K}_*^{MW}(R/p,\omega_{(R/p)/k}) \arrow[r, "\sum_p (m_p)_\epsilon\mathrm{Tr}_{\psi_p(a_1),\dots,\psi_p(a_r)/L}"'] & \mathbf{K}_*^{MW}(L,\omega_{L/k})
\end{tikzcd}$$

is commutative where m_p is the length of the localized ring $R_{(p)}$.

Proof. We prove the theorem by induction. For $r = 1$, this is Lemma 3.6. Write $E(x_1) \otimes_E L = \prod_j R_j$ for some Artin local L-algebras R_j, and decompose the finite dimensional L-algebra $F \otimes_{E(x_1)} R_j$ as $F \otimes_{E(x_1)} R_j = \prod_i R_{ij}$ for some local L-algebras R_{ij}. We have $F \otimes_E L \simeq \prod_{i,j} R_{ij}$. Denote by L_j (resp. L_{ij}) the residue fields of the Artin local L-algebras R_j (resp. R_{ij}), and m_j (resp. m_{ij}) for their geometric multiplicity. We can conclude as the following diagram commutes

$$\begin{tikzcd}
\mathbf{K}_*^{\mathrm{MW}}(F, \omega_{F/k}) \arrow[r, "\mathrm{Tr}_{x_1,\ldots,x_r/E}"] \arrow[d, "\oplus_{ij} \mathrm{res}_{L_{ij}/F}"'] & \mathbf{K}_*^{\mathrm{MW}}(E(x_1), \omega_{E(x_1)/k}) \arrow[r, "\mathrm{Tr}_{x_1/E}"] \arrow[d, "\oplus \mathrm{res}_{L_j/E(x_1)}"] & \mathbf{K}_*^{\mathrm{MW}}(E, \omega_{E/k}) \arrow[d, "\mathrm{res}_{L/E}"] \\
\bigoplus_{ij} \mathbf{K}_*^{\mathrm{MW}}(L_{ij}, \omega_{L_{ij}/k}) \arrow[r, "\sum_{ij}(m_{ij}m_j^{-1})_\epsilon \mathrm{Tr}_{\psi_{ij}(x_1),\ldots,\psi_{ij}(x_r)/L_j}"'] & \bigoplus_j \mathbf{K}_*^{\mathrm{MW}}(L_j, \omega_{L_j/k}) \arrow[r, "\sum_j (m_j)_\epsilon \mathrm{Tr}_{\psi_j(x_1)/L}"'] & \mathbf{K}_*^{\mathrm{MW}}(L, \omega_{L/k})
\end{tikzcd}$$

since both squares are commutative by the inductive hypothesis, the case $r = 1$, and the multiplicity formula $(mn)_\epsilon = m_\epsilon n_\epsilon$ for any natural numbers m, n. $\square$

THEOREM 3.9. *Assume that Theorem 2.17 holds for all p-primary fields E for any prime number p. Then the theorem holds for any field E.*

Proof. Consider two decompositions

$$E \subset E(x_1) \subset E(x_1, x_2) \subset \cdots \subset E(x_1, \ldots, x_r) = F.$$

and

$$E \subset E(y_1) \subset E(y_1, y_2) \subset \cdots \subset E(y_1, \ldots, y_s) = F.$$

of F. Let $\alpha \in \mathbf{K}_*^{\mathrm{MW}}(F)$ and denote by δ the element $\mathrm{Tr}_{x_1,\ldots,x_r/E}(\alpha) - \mathrm{Tr}_{y_1,\ldots y_s/E}(\alpha)$. Fix p a prime number and let L be a maximal prime to p extension of E (L has no nontrivial finite extension of degree prime to p). With the notation of Theorem 3.8, the map $\sum_{\mathfrak{p}} (m_p)_\epsilon \mathrm{Tr}_{\psi_{\mathfrak{p}}(x_1),\ldots,\psi_{\mathfrak{p}}(x_r)/L}$ does not depend on the choice of x_i according to the assumption. Hence $\mathrm{res}_{L/E}(\delta) = 0$ and we can find a finite extension L_p/E of degree prime to p such that $\mathrm{res}_{L_p/E}(\delta) = 0$. Since this is true for all prime number p, we see that the assumption of Lemma 3.4 are satisfied. Thus $\delta = 0$ and the theorem is proved. $\square$

3.2 Proof in the p-primary case

PROPOSITION 3.10 (Bass-Tate-Morel Lemma). *Let $F(x)$ be a monogenic extension of F. Then $\mathbf{K}_*^{MW}(F(x))$ is generated as a left $\mathbf{K}_*^{MW}(F)$-module by elements of the form*

$$\boldsymbol{\eta}^m \cdot [p_1(x), p_2(x), \ldots, p_n(x)]$$

where the p_i are monic irreducible polynomials of $F[t]$ satisfying

$$\deg(p_1) < \deg(p_2) < \cdots < \deg(p_n) \leq d - 1$$

where d is the degree of the extension $F(x)/F$.

Proof. Straightforward computations (see also [Mor12, Theorem 3.24 and Lemma 3.25.1]). □

Corollary 3.11. *Let F/E be a finite field extension and assume one of the following conditions holds:*

- *F/E is a quadratic extension,*
- *F/E is a prime degree p extension and E has no nontrivial extension of degree prime to p.*

Then $\mathbf{K}_*^{MW}(F)$ is generated as a left $\mathbf{K}_*^{MW}(E)$-module by $F^\times$.

Proof. In both cases, the extension F/E is simple and the only monic irreducible polynomial in $E[t]$ of degree strictly smaller than $[F:E]$ are the polynomials of degree 1. We conclude by Proposition 3.10. □

In the following, we fix a prime number p and E a p-primary field.

Proposition 3.12. *Let $F = E(x)$ be a monogenic extension of E of degree p. Then the transfers $\mathrm{Tr}_{x/E} : \mathbf{K}_*^{MW}(E(x), \omega_{E(x)/k}) \to \mathbf{K}_*^{MW}(E, \omega_{E/k})$ do not depend on the choice of x.*

Proof. According to Lemma 3.11, the group $\mathbf{K}_*^{\mathrm{MW}}(F, \omega_{F/k})$ is generated by products of the form $\mathrm{res}_{F/E}(\alpha) \cdot [\beta]$ with $\alpha \in \mathbf{K}_*^{\mathrm{MW}}(E, \omega_{E/k})$ and $\beta \in F^\times$. According to 2.14, we have the projection formula

$$\mathrm{Tr}_{x/E}(\mathrm{res}_{F/E}(\alpha) \cdot [\beta]) = \alpha \cdot \mathrm{Tr}_{F/E}([\beta]).$$

It remains to prove that the right-hand side $Tr_{F/E}([\beta]) \in \mathbf{K}_1^{\mathrm{MW}}(E)$ does not depend on a choice of x. For that, we consider the Cartesian square

$$\begin{tikzcd} \mathbf{K}_1^{\mathrm{MW}}(E) \arrow[r] \arrow[d] & \mathbf{I}(E) \arrow[d] \\ \mathbf{K}_1^{\mathrm{M}}(E) \arrow[r] & \mathbf{I}(E)/\mathbf{I}^2(E) \end{tikzcd}$$

where $\mathbf{K}_1^{\mathrm{M}}(E) = E^\times$ and $\mathbf{I}(E)$ is the fundamental ideal (we refer to the proof of [Fas20, Theorem 1.4] with the following remarks: *ibid.* assume the characteristic to be different from 2 but this is not necessary according to [Mor12, Remark 3.12]; the proof of *ibid.* uses Voevodsky's affirmation of Milnor conjecture but this is not needed in our case since we only work in degree $n = 1$).

By naturality of the previous square, we are reduced to proving the independence of the transfer map for the two cases $\mathbf{K}_1^{\mathrm{M}}(E) = E^\times$ and $\mathbf{I}(E)$ which we assume to have elementary proofs somewhere in the literature (recall that a non-elementary proof of Proposition 3.12 can also be found in [Mor12, Chapter 5]). Indeed, for $\mathbf{K}_1^{\mathrm{M}}(E) = E^\times$, the transfer map is nothing but the Galois norm; for $\mathbf{I}(E)$, we only have to consider the two cases where $E(x)/E$ is a purely inseparable extension or a separable extension, which are true in characteristic $\neq 2$ according to [Fas08, Lemma 6.4.6] and [Fas20, Example 1.23] (in characteristic 2, we believe the work of Fasel on Witt transfers could be extended but, for the present article, we simply refer to the book of Morel, e.g. [Mor12, Proof of Corollary 5.2 in the case $r = 1$]). $\square$

REMARK 3.13. We may now use the notation $\mathrm{Tr}_{F/E} : \mathbf{K}_*^{\mathrm{MW}}(F, \omega_{F/k}) \to \mathbf{K}_*^{\mathrm{MW}}(E, \omega_{E/k})$ if F/E is a field extension of prime degree p.

PROPOSITION 3.14. *Let F be a field complete with respect to a discrete valuation v, and F'/F a normal extension of degree p. Denote by v' the unique extension of v to F'. Then the diagram*

$$\begin{array}{ccc} \mathbf{K}_*^{MW}(F', \omega_{F'/k}) & \xrightarrow{\partial_{v'}} & \mathbf{K}_{*-1}^{MW}(\kappa(v'), \omega_{\kappa(v')}) \\ \downarrow {\scriptstyle \mathrm{Tr}_{F'/F}} & & \downarrow {\scriptstyle \mathrm{Tr}_{\kappa(v')/\kappa(v)}} \\ \mathbf{K}_*^{MW}(F) & \xrightarrow[\partial_v]{} & \mathbf{K}_{*-1}^{MW}(\kappa(v)) \end{array}$$

is commutative.

Proof. This is a particular case of [Mor12, Remark 5.20] (the completeness assumption is not needed). We note that the proof of [Mor12, Remark 5.20] does not depend on [Mor12, Lemma 5.5] (thus there is no loophole in the proof of Theorem 2.17). Moreover, even though the proof of [Mor12, Remark 5.20] is only three pages long, we hope that the completeness assumption could lead to a shorter proof. $\square$

COROLLARY 3.15. *Let F/E be a normal extension of degree p and let $x \in (\mathbb{A}^1_E)^{(1)}$. Then the diagram*

$$\begin{array}{ccc} \mathbf{K}_*^{MW}(F(t), \omega_{F(t)/k}) & \xrightarrow{\oplus \partial_y} & \bigoplus_{y \mapsto x} \mathbf{K}_{*-1}^{MW}(\kappa(y), \omega_{\kappa(y)/k}) \\ \downarrow {\scriptstyle \mathrm{Tr}_{F(t)/E(t)}} & & \downarrow {\scriptstyle \sum \mathrm{Tr}_{\kappa(y)/\kappa(x)}} \\ \mathbf{K}_*^{MW}(E(t), \omega_{E(t)/k}) & \xrightarrow[\partial_x]{} & \mathbf{K}_{*-1}^{MW}(\kappa(x), \omega_{\kappa(x)/k}) \end{array}$$

is commutative, where $y \to x$ denotes the set of elements $y \in (\mathbb{A}^1_F)^{(1)}$ mapping to x through the canonical morphism.

Proof. Denote by $\hat{E}_x$ (resp. $\hat{F}_y$) the completions of $E(t)$ (resp. $F(t)$) with respect to the valuations defined by x (resp. y). Consider the following diagram

$$\begin{tikzcd}
\mathbf{K}_*^{\mathrm{MW}}(F(t),\omega_{F(t)/k}) \arrow[r] \arrow[d, "\mathrm{Tr}_{F(t)/E(t)}"'] & \bigoplus_{y\to x}\mathbf{K}_*^{\mathrm{MW}}(\hat{F}_y,\omega_{\hat{F}_y/k}) \arrow[r, "\oplus\partial_y"] \arrow[d, "\sum \mathrm{Tr}_{\hat{F}_y/\hat{F}_x}"] & \bigoplus_{y\to x}\mathbf{K}_{*-1}^{\mathrm{MW}}(\kappa(y),\omega_{\kappa(y)/k}) \arrow[d, "\sum \mathrm{Tr}_{\kappa(y)/\kappa(x)}"] \\
\mathbf{K}_*^{\mathrm{MW}}(E(t),\omega_{E(t)/k}) \arrow[r] & \mathbf{K}_*^{\mathrm{MW}}(\hat{E}_x,\omega_{\hat{E}_x/k}) \arrow[r, "\partial_x"] & \mathbf{K}_{*-1}^{\mathrm{MW}}(\kappa(x),\omega_{\kappa(x)/k}).
\end{tikzcd}$$

The left-hand square is commutative according to Theorem 3.8. The right-hand square commutes according to Proposition 3.14. Hence the corollary. □

LEMMA 3.16. *Let L/E be a normal extension of degree p, and let $E(a)/E$ be a monogenic finite extension. Assume that L and $E(a)$ are both subfields of some algebraic extension of E, and denote by $L(a)$ their composite. Then the following diagram*

$$\begin{tikzcd}
\mathbf{K}_*^{MW}(L(a),\omega_{L(a)/k}) \arrow[r, "\mathrm{Tr}_{a/L}"] \arrow[d, "\mathrm{Tr}_{L(a)/E(a)}"'] & \mathbf{K}_*^{MW}(L,\omega_{L/k}) \arrow[d, "\mathrm{Tr}_{L/E}"] \\
\mathbf{K}_*^{MW}(E(a),\omega_{E(a)/k}) \arrow[r, "\mathrm{Tr}_{a/E}"'] & \mathbf{K}_*^{MW}(E,\omega_{E/k})
\end{tikzcd}$$

is commutative.

Proof. First of all, we note that the vertical maps are independent of choices by Lemma 3.12 (note that if $L=E(a)$, then $\mathrm{Tr}_{L(a)/E(a)}=\mathrm{Id}$ does not depend on any choices). Let x (resp y_0) be the closed point of $\mathbb{A}^1_E$ (resp. $\mathbb{A}^1_L$) defined by the minimal polynomial of a over E (resp. L). Given $\alpha\in\mathbf{K}_*^{\mathrm{MW}}(L(a),\omega_{L(a)/k})$, we have $\mathrm{Tr}_{a/L}(\alpha)=-\partial_\infty(\beta)$ for some $\beta\in\mathbf{K}_{*+1}^{\mathrm{MW}}(L(t),\omega_{L(t)/k})$ satisfying $\partial_{y_0}(\beta)=\alpha$ and $\partial_y(\beta)=0$ for $y\neq y_0$. By Corollary 3.15

$$\partial_x(\mathrm{Tr}_{L(t)/E(t)}(\beta))=\textstyle\sum_{y\mapsto x}\mathrm{Tr}_{\kappa(y)/\kappa(x)}(\partial_y(\beta))=\mathrm{Tr}_{\kappa(y_0)/\kappa(x)}(\alpha),$$

and, similarly, $\partial_{x'}(\mathrm{Tr}_{L(t)/E(t)}(\beta))=0$ for $x\neq x'$. Hence by definition of the transfer map $\mathrm{Tr}_{a/E}$ we have

$$\mathrm{Tr}_{a/E}(\mathrm{Tr}_{L(a)/E(a)}(\alpha))=-\partial_\infty(\mathrm{Tr}_{L(t)/E(t)}(\beta)).$$

Moreover, since the only point of $\mathbb{P}^1_L$ above ∞ is ∞, another application of Corollary 3.15 gives

$$\partial_\infty(\mathrm{Tr}_{L(t)/E(t)}(\beta))=\mathrm{Tr}_{L/E}(\partial_\infty(\beta)).$$

Hence the result.

$$\mathrm{Tr}_{a/E}(\mathrm{Tr}_{L(a)/E(a)}(\alpha))=-\mathrm{Tr}_{L/E}(\partial_\infty(\beta))=\mathrm{Tr}_{L/E}(\mathrm{Tr}_{a/L}(\alpha)).$$

□

Proof of Theorem 2.17. We keep the previous notations. We already know that it suffices to treat the case when E has no nontrivial extension of degree prime to p (according to Theorem 3.9). Let p^m be the degree of the extension F/E. We prove the result by induction on m. The case $m = 1$ follows from Proposition 3.12. Consider two decompositions

$$E \subset E(x_1) \subset E(x_1, x_2) \subset \cdots \subset E(x_1, \ldots, x_r) = F.$$

and

$$E \subset E(y_1) \subset E(y_1, y_2) \subset \cdots \subset E(y_1, \ldots, y_s) = F.$$

of F. By Lemma 2.5, the extension $E(x_1)/E$ contains a normal subfield $E(x_1')$ of degree p over E. Applying Lemma 3.16 with $a = x_1$ and $L = E(x_1')$ yields $\mathrm{Tr}_{x_1/E} = \mathrm{Tr}_{x_1'/E} \circ \mathrm{Tr}_{x_1/E(x_1')}$. Hence, without loss of generality, we may assume that $x_1 = x_1'$ and, similarly, $[E(y_1) : E] = p$. Write F_0 for the composite of the fields $E(x_1)$ and $E(y_1)$ in F and write $F = F_0(z_1, \ldots, z_t)$ with $z_i \in F$. The fields $E(x_1)$ and $E(y_1)$ have no nontrivial prime to p extension, thus we may conclude by the induction hypothesis that the triangles

$$\begin{tikzcd}
\mathbf{K}^{\mathrm{MW}}_*(F, \omega_{F/k}) \arrow[r, "\mathrm{Tr}_{x_2,\ldots,x_r/E(x_1)}"] \arrow[d, "\mathrm{Tr}_{z_1,\ldots,z_t/F_0}"'] & \mathbf{K}^{\mathrm{MW}}_*(E(x_1), \omega_{E(x_1)/k}) \\
\mathbf{K}^{\mathrm{MW}}_*(F_0, \omega_{F_0/k}) \arrow[ur, "\mathrm{Tr}_{F_0/E(x_1)}"'] &
\end{tikzcd}$$

and

$$\begin{tikzcd}
\mathbf{K}^{\mathrm{MW}}_*(F, \omega_{F/k}) \arrow[r, "\mathrm{Tr}_{y_2,\ldots,y_s/E(y_1)}"] \arrow[d, "\mathrm{Tr}_{z_1,\ldots,z_t/F_0}"'] & \mathbf{K}^{\mathrm{MW}}_*(E(y_1), \omega_{E(y_1)/k}) \\
\mathbf{K}^{\mathrm{MW}}_*(F_0, \omega_{F_0/k}) \arrow[ur, "\mathrm{Tr}_{F_0/E(y_1)}"'] &
\end{tikzcd}$$

are commutative.

Moreover, Lemma 3.16 for $a = x_1$ and $L = E(y_1)$ implies that the following diagram

$$\begin{tikzcd}
\mathbf{K}^{\mathrm{MW}}_*(F_0, \omega_{F_0/k}) \arrow[r, "\mathrm{Tr}_{F_0/E(x_1)}"] \arrow[d, "\mathrm{Tr}_{F_0/E(y_1)}"'] & \mathbf{K}^{\mathrm{MW}}_*(E(x_1), \omega_{E(x_1)/k}) \arrow[d, "\mathrm{Tr}_{x_1/E}"] \\
\mathbf{K}^{\mathrm{MW}}_*(E(y_1), \omega_{E(y_1)/k}) \arrow[r, "\mathrm{Tr}_{y_1/E}"'] & \mathbf{K}^{\mathrm{MW}}_*(E, \omega_{E/k})
\end{tikzcd}$$

is commutative. Putting everything together, we conclude that $\mathrm{Tr}_{x_1,\ldots,x_r/E} = \mathrm{Tr}_{y_1,\ldots,y_s/E}$.

$\square$

3.3 Applications in motivic homotopy theory

We end this section with a discussion of a conjecture of Morel in motivic homotopy theory. Milnor-Witt K-theory is a fundamental object in motivic homotopy theory since it computes the homotopy groups of spheres (in the sense of [Mor12, Chapter 6]). Moreover, Milnor-Witt K-theory is a particular case of the notion of *homotopy sheaf* as defined below.

3.17. Consider $M \in \mathbf{HI}(k)$ a homotopy sheaf, i.e. a Nisnevich sheaf over the category of smooth k-schemes $\mathbf{Sm}_k$ with value in the category of abelian groups and satisfying the following property (*strong $\mathbb{A}^1$-invariance*): for any smooth scheme X, the map

$$H^i(X, M) \to H^i(\mathbb{A}^1_X, M)$$

of Nisnevich sheaf cohomology groups induced by the canonical projection $\mathbb{A}^1_X \to X$ is a bijection for $i \in \{0, 1\}$.

For instance, the Milnor-Witt K-theory $\mathbf{K}_n^{\mathrm{MW}}$ in degree n defines a homotopy sheaf (for any fixed integer n).

Recall that the contraction of M is the sheaf defined by

$$X \mapsto \ker(M(\mathbb{G}_m \times X) \to M(X))$$

and is denoted by M_{-1}; this is again a homotopy sheaf. Moreover, M_{-1} has a structure of GW-module and, for any valued field (F, v), we have a (twisted) residue map

$$M(F) \to M(\kappa(v), \omega_v) := M(\kappa(v)) \otimes_{\mathbb{Z}[\kappa(v)^\times]} \mathbb{Z}[\omega_v^\times].$$

3.18. Let M be a homotopy sheaf and M_{-1} its contraction. We recall the construction of the Bass-Tate transfer maps

$$\mathrm{Tr}_\psi = \mathrm{Tr}_{F/E} : M_{-1}(F, \omega_{F/k}) \to M_{-1}(E, \omega_{E/k})$$

defined for any finite map $\psi : E \to F$ of fields.

THEOREM 3.19. *Let $M \in \mathbf{HI}(k)$ be a homotopy sheaf. Let F be a field and $F(t)$ the field of rational functions with coefficients in F in one variable t. We have a split short exact sequence*

$$0 \to M(F) \xrightarrow{\mathrm{res}} M(F(t)) \xrightarrow{d} \bigoplus_{x \in (\mathbb{A}^1_F)^{(1)}} M_{-1}(\kappa(x), \omega_x) \to 0$$

where $d = \bigoplus_{x \in (\mathbb{A}^1_F)^{(1)}} \partial_x$ is the usual differential (see [Mor12, Chapter 4]).

Proof. See [Mor12, §4.2, page 97] and [Mor12, Theorem 5.38]. □

DEFINITION 3.20 (Coresidue maps). Keeping the previous notations, the fact that the homotopy sequence is split allows us to define *coresidue maps*

$$\rho_x : M_{-1}(\kappa(x), \omega_x) \to M(F(t))$$

for any closed points $x \in (\mathbb{A}^1_F)^{(1)}$, satisfying $\partial_x \circ \rho_x = \mathrm{Id}_{\kappa(x)}$ and $\partial_y \circ \rho_x = 0$ for $x \neq y$ where $y \in (\mathbb{A}^1_F)^{(1)}$.

Definition 3.21 (Bass-Tate transfers). Let $M \in \mathbf{HI}(k)$ be a homotopy sheaf. Let F be a field and $F(t)$ the field of rational functions with coefficients in F in one variable t. For $x \in (\mathbb{A}^1_F)^{(1)}$, we define the Bass-Tate transfer

$$\mathrm{Tr}_{x/F} : M_{-1}(F(x), \omega_{F(x)/k}) \to M_{-1}(F, \omega_{F/k})$$

by the formula $\mathrm{Tr}_{x/F} = -\partial_\infty \circ \rho_x$.

Remark 3.22. There is also an equivalent definition of the Bass-Tate transfers that does not use the coresidue maps (see [Mor12, §4.2]).

Definition 3.23. Let $F = E(x_1, x_2, \ldots, x_r)$ be a finite extension of a field E and consider the chain of subfields

$$E \subset E(x_1) \subset E(x_1, x_2) \subset \cdots \subset E(x_1, \ldots, x_r) = F.$$

Define by induction:

$$\mathrm{Tr}_{x_1,\ldots,x_r/E} = \mathrm{Tr}_{x_r/E(x_1,\ldots,x_{r-1})} \circ \cdots \circ \mathrm{Tr}_{x_2/E(x_1)} \circ \mathrm{Tr}_{x_1/E}$$

Conjecture 3.24 (Morel conjecture)**.** *Let $F = E(x_1, \ldots, x_r)/E$ be a finite field extension. Then the map*

$$\mathrm{Tr}_{x_1,\ldots,x_r/E} : M_{-1}(F, \omega_{F/k}) \to M_{-1}(E, \omega_{E/k})$$

does not depend on the choice of the generating system $(x_1, \ldots, x_r)$.

Remark 3.25.

1. This was claimed by Morel in [Mor12, Remark 4.31] and [Mor11, Remark 5.10] (see also [Bac20, Remark 4.3] for a similar conjecture).

2. Morel proved in [Mor12, Chapter 4] that the conjecture is true if the contracted homotopy sheaf M_{-1} is replaced by M_{-2}. The proof of Morel uses in a fundamental way the cohomology group $H^2((\mathbb{P}^1)^2, M_{-2})$ and cannot be easily applied to prove the conjecture in full generality.

3. In [Fel20b, Theorem 6.1.6], the author proved that, if M is a homotopy sheaf, then Conjecture 3.24 is true if and only if M has a structure of Milnor-Witt transfers (or, equivalently, a structure of framed transfers).

4. We also know that the conjecture is true in full generality if we work with rational homotopy sheaves $M_{\mathbb{Q},-1}$ (see [Fel20b, Theorem 4.1.19]).

Following the ideas of the previous section, we can reduce the conjecture to the case of p-primary fields.

THEOREM 3.26. *In order to prove Conjecture 3.24 (i.e. a contracted homotopy sheaf M_{-1} has functorial transfers), it suffices to consider the case of p-primary fields (where p is a prime number).*

Proof. We can use verbatim the proof of Theorem 3.9 where Theorem 3.8 is replaced by [Fel20b, Theorem 4.1.16] and Lemma 3.4 still applies thanks to more general projection formulas [Fel20b, Theorem 4.1.15]. □

We still have hope to prove the conjecture in full generality with the help of the previous theorem.

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