Problems and exercises in motivic homotopy theory
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1 Milnor K-theory

The Milnor K-groups $K_n^M(k)$ attached to a field k is the quotient of the n-th tensor power $(k^\times)^{\otimes n}$ of the multiplicative group of k by the subgroup generated by those elements $a_1 \otimes \cdots \otimes a_n$ for which $a_i + a_j = 1$ for some $1 \leq i < j \leq n$. Thus $K_0^M(k) = \mathbb{Z}$ and $K_1^M(k) = k^\times$. Elements of $K_n^M(k)$ are called *symbols*; we write $[a_1, \ldots, a_n]$ for the image of $a_1 \otimes \cdots \otimes a_n$ in $K_n^M(k)$.

1. Show that Milnor K-groups are functorial with respect to field extensions: given an inclusion $\varphi : k \subset K$, there is a natural map $i_{K/k} : K_n^M(k) \to K_n^M(K)$ induced by φ.

   Given $\alpha \in K_n^M(K)$, we shall often abbreviate $i_{K/k}(\alpha)$ by α_K.

2. Show that the product pairings

   $$(k^\times)^{n\otimes} \times (k^\times)^{m\otimes}$$

   induce a structure of graded ring on

   $$K_*^M(k) = \bigoplus\nolimits_{n\geq 0} K_n^M(k).$$

3. (a) Prove that the group $K_2^M(k)$ satisfies the relations

   $$[x, -x] = 0 \text{ and } [x, x] = [x, -1].$$

   (b) Prove that the product operation on $K_*^M(k)$ is graded-commutative, i.e. it satisfies

   $$[\alpha, \beta] = (-1)^{nm}[\beta, \alpha]$$

   for $\alpha \in K_n^M(k)$ and $\beta \in K_m^M(k)$

4. Let $\mathbf{F}$ be a finite field. Prove that, for all $n > 1$, the groups $K_n^M(\mathbf{F})$ are trivial.

5. Let K be a field equipped with a discrete valuation $v : K^\times \to \mathbb{Z}$. Denote by $\mathcal{O}_v$ the associated valuation ring and by $\kappa(v)$ its residue field.

   (a) Fix π a local parameter (i.e. an element satisfying $v(\pi) = 1$). For n a natural number, show that $K_n^M(K)$ is generated by symbols of the form $[\pi, u_2, \dots, u_n]$ and $[u_1, \dots, u_n]$ where u_i are units in $\mathcal{O}_v$.

   (b) For each $n > 0$, there exists a unique morphism
   $$\partial^M : K_n^M(K) \to K_{n-1}^M(\kappa(v))$$
   satisfying
   $$\partial^M([\pi, u_2, \dots, u_n]) = [\bar{u_2}, \dots, \bar{u_n})$$
   for all local parameters π and all units u_i, where $\bar{u_i}$ denotes the image of u_i in $\kappa(v)$.
   Moreover, once a local parameter π is fixed, there is a unique morphism
   $$s_\pi^M : K_n^M(K) \to K_n^M(\kappa(v))$$
   with the property
   $$s_\pi^M([\pi^{i_1}u_1, \dots, \pi^{i_n}u_n]) = [\bar{u_1}, \dots, \bar{u_n}]$$
   for all integers i_j and units u_i of $\mathcal{O}_v$.

   (c) Prove that the tame symbol $\partial^M : K_1^M(K) \to K_0(\kappa(v))$ is the valuation map $v : K^\times \to \mathbb{Z}$, and that the tame symbol $\partial^M : K_2^M(K) \to K_1^M(\kappa(v))$ is given by the formula
   $$\partial^M([a, b]) = (-1)^{v(a)v(b)}\overline{a^{v(b)}b^{-v(a)}}$$
   where the lines denotes the image in $\kappa(v)$.

   (d) Prove that, for $[a_1, \dots, a_n] \in K_n^M(K)$, one has the formula
   $$s_\pi^M([a_1, \dots, a_n]) = \partial^M([-\pi, a_1, \dots, a_n])$$
   for all local parameters π.

   (e) Let L/K be a field extension and b_L a discrete valuation of L extending v with residue field $\kappa(v_L)$ and ramification index e. Denoting the associated tame symbol by ∂_L^M, one has for all $\alpha \in K_n^M(K)$
   $$\partial_L^M(\alpha_L) = e \cdot \partial^M(\alpha).$$

   (f) Denote by U_n the subgroup of $K_n^M(K)$ generated by those symbols $[u_1, \dots u_n]$ where all the u_i are units in $\mathcal{O}_v$, and $U_n^1 \subset K_n^M(K)$ the subgroup generated by symbols $[x_1, \dots, x_n]$ with x_1 a unit in $\mathcal{O}_v$ satisfying $\overline{x}_1 = 1$.

i. Prove that $U_n^1 \subset U_n$.

ii. Prove that we have exact sequences

$$0 \longrightarrow U_n \longrightarrow K_n^M(K) \xrightarrow{\partial^M} K_{n-1}^M(\kappa(v)) \longrightarrow 0$$

and

$$0 \longrightarrow U_n^1 \longrightarrow K_n^M(K) \xrightarrow{(s_\pi^M, \partial^M)} K_n^M(\kappa(v)) \oplus K_{n-1}^M(\kappa(v)) \longrightarrow 0.$$

(g) Assume moreover that K is complete with respect to v, and let $m > 0$ be an integer invertible in $\kappa(v)$.
Prove that the pair (s_π^M, ∂^M) induces an isomorphism

$$K_n^M(K)/mK_n^M(K) \simeq$$
$$K_n^M(\kappa(v))/mK_n^M(\kappa(v)) \oplus K_{n-1}^M(\kappa(v))/mK_{n-1}^M(\kappa(v)).$$

6. Recall that the discrete valuations of $k(t)$ trivial on k correspond to the local rings of closed points P on the projective line $\mathbb{P}_k^1$. As before, we denote by $\kappa(P)$ their residue fields and by v_P the associated valuations. At each closed point $P \neq \infty$ a local parameter is furnished by a monic irreducible polynomial $\pi_P \in k[t]$; at $P = \infty$ one may take $\pi_P = t^{-1}$. The degree of the field extension $[\kappa(P), k]$ is called the degree of the closed point P; it equals the degree of the polynomial π_P. Thus we obtain tame symbols

$$\partial_P^M : K_n^M(k(t)) \to K_{n-1}^M(\kappa(P))$$

and specialization maps

$$s_\pi^M : K_n^M(k(t)) \to K_n^M(\kappa(P)).$$

(a) Show that the image of the product map

$$\partial^M := (\partial_P^M) : K_n^M(k(t)) \to \prod_{P \in \mathbb{P}^1 - \{\infty\}} K_{n-1}^M(\kappa(P))$$

lies in the direct sum.

(b) Denote by L_d the subgroup of $K_n^M(k(t))$ generated by those symbols $[f_1, \ldots, f_n]$ where f_i are polynomials in $k[t]$ of degree $\leq d$. For each $d > 0$, consider the map

$$\partial_d^M : K_n^M(k(t)) \to \bigoplus_{\deg(P)=d} K_{n-1}^M(\kappa(P))$$

defined as the direct sum of the maps ∂_P^M for all closed points P of degree d.
Prove that its restriction to L_d induces an isomorphism

$$\overline{\partial}_d^M : L_d/L_{d-1} \simeq \bigoplus_{\deg(P)=d} K_{n-1}^M(\kappa(P)).$$

(c) (*Homotopy invariance*) Prove that the sequence

$$0 \longrightarrow K_n^M(k) \longrightarrow K_n^M(k(t)) \xrightarrow{\partial^M} \bigoplus_{P\in\mathbb{P}^1-\{\infty\}} K_{n-1}^M(\kappa(P)) \longrightarrow 0$$

is exact and split by the specialization map $s_{t^{-1}}^M$ at ∞.

2 Milnor-Witt K-theory

1. Generalize the previous results to the Milnor-Witt K-groups $\mathbf{K}_*^{\mathrm{MW}}(k)$.

3 Smooth models

1. Let E be a finitely generated field over the perfect field k. By definition, a *smooth model* of E is an affine smooth scheme $X = \operatorname{Spec} A$ of finite type such that A is a sub-k-algebra of E, with function field E.

   Convince yourself that such a smooth model always exists.

2. Let E/k and L/k be two extensions and $\varphi : E \to L$ a morphism such that the extension L/E is finite. By definition, we call *k-model of L/E* any triplet $((X, x), (Y, y), f : Y \to X)$ such that (X, x) is a model of E/k, (Y, y) is a model of L/k and f is a dominant finite morphism making the following diagram commutative:

$$\begin{tikzcd} \operatorname{Spec} L \arrow[r, "\operatorname{Spec}\varphi"] \arrow[d, "y"'] & \operatorname{Spec} E \arrow[d, "x"] \\ Y \arrow[r, "f"] & X \end{tikzcd}$$

   where the vertical maps are induced by the points x and y.

   (a) Let $f : Y \to X$ be an equidimensional finite morphism of schemes. Assume that U is a dense open subscheme of Y.

   Prove that the open subscheme $f^{-1}(X - f(Y - U))$ is dense containing U.

   (b) Let E/k be an extension and E/L a finite extension of fields.

   Prove that there exists a k-model of L/E.

3. Let E/k be an extension and L/E a finite extension. Consider $f : Y \to X$ and $f' : Y' \to X'$ two k-models of L/E.

   Prove that there is a k-model $f'' : Y'' \to X''$ of L/E such that the diagram

$$
\begin{tikzcd}
Y \arrow[r, "f"] & X \\
Y'' \arrow[r, "f''"] \arrow[u] \arrow[d] & X'' \arrow[u] \arrow[d] \\
Y' \arrow[r, "f'"] & X'
\end{tikzcd}
$$

is commutative and compatible with the base points.

4 Grothendieck-Witt groups

1. Let E be a field of characteristic $p > 0$. Let $\alpha \in \mathrm{GW}(E)$ be an element in the kernel of the rank morphism $\mathrm{GW}(E) \to \mathbb{Z}$.

   Prove that α is nilpotent in $\mathrm{GW}(E)$.

5 Enumerative geometry

5.1 Apollonius circles

1. Show that the two following definitions are equivalent:

   (a) A circle in $\mathbb{P}^2$ is given by the equation
   $$(x - az)^2 + (y - bz)^2 = r^2 z^2.$$
   (b) A circle in $\mathbb{P}^2$ is a conic given by $V(f)$ where $f \in (z, x^2 + y^2)$.

2. Define
   $$\Phi = \{(r, C) \in D \times \mathbb{P}^3 \,|\, C \text{ is tangent to } D \text{ at } r\}$$
   where D is a smooth circle and $\mathbb{P}^3$ is viewed as the space of circles.

   Prove that the correspondence Φ is 2-dimensional and irreducible.

3. Denote by $\pi_2 : \Phi \to \mathbb{P}^3$ the second canonical projection and $Z_D = \pi_2(\Phi)$ its image.

   Prove that Z_D has dimension 2.

4. Consider a line L inside $\mathbb{P}^3$. Viewing $\mathbb{P}^3$ again as the space of circles, L parameterizes a family of circles $\{C_t\}_{t \in \mathbb{P}^3}$.

   Assuming L is generic, prove that $L \cap Z_D$ consists of 2 points.

   Conclude that Z_D is a quadric surface.

5. Let C be a circle tangent to D. Prove that the line between C and D is in Z_D. Hence Z_D is a quadric cone with vertex in D.

6. Given three circles in general position, how many circles are tangent to all three?

References