

Milnor-Witt homotopy sheaves and Morel generalized transfers

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Abstract

We explore a conjecture of Morel about the Bass-Tate transfers defined on the contraction of a homotopy sheaf and prove that the conjecture is true with rational coefficients. Moreover, we study the relations between (contracted) homotopy sheaves, sheaves with Morel generalized transfers and Milnor-Witt homotopy sheaves, and prove an equivalence of categories. As applications, we describe the essential image of the canonical functor that forgets Milnor-Witt transfers and use these results to discuss the conservativity conjecture in motivic homotopy theory due to Bachmann and Yakerson.

Keywords: Motivic homotopy theory, Homotopy sheaves, Geometric transfers, Milnor-Witt K-theory

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1. Introduction

1.1. Current work

In Morel (2012), Morel studied homotopy invariant Nisnevich sheaves in order to provide computational tools in \mathbb{A}^1 -homotopy analogous to Voevodsky's theory of sheaves with transfers. The most basic result is that (unramified) sheaves are characterized by their sections on fields and some extra data (see Subsection 2.1). One of the main theorem of Morel (2012) is the equivalence between the notions of *strongly* \mathbb{A}^1 -invariance and *strictly* \mathbb{A}^1 -invariance for sheaves of abelian groups (see *loc. cit.* Theorem 1.16). In order to prove this, Morel defined geometric transfers on the contraction M_{-1} of a homotopy sheaf (i.e. a strongly \mathbb{A}^1 -invariant Nisnevich sheaf of abelian groups). The definition is an adaptation of the original one of Bass and Tate for Milnor K-theory Bass and Tate (1973). Morel proved that the transfers are functorial (i.e. they do not depend on the choice of generators) for any two-fold contraction M_{-2} of a homotopy sheaf and conjectured that the result should hold for M_{-1} (see Conjecture 4.1.13 or (Morel, 2012, Remark 4.31)).

The notion of *sheaves with generalized transfers* was first defined in (Morel, 2011, Definition 5.7) as a way to formalize the different structures naturally arising on some homotopy sheaves. In Section 3, we give a slightly modified definition of *sheaves with generalized transfers* which takes into account twists by the usual line bundles. Following (Morel, 2012, Chapter 5), we define the Rost-Schmid complex associated to such homotopy sheaves and study the usual pushforward maps f_* , pullback maps g^* , GW-action $\langle a \rangle$ and residue maps ∂ . Moreover, we prove the following theorem.

Theorem 1 (see **Theorem 3.2.4**). *Let $M \in \mathbf{HI}^{\text{gtr}}(k)$ be a homotopy sheaf with generalized transfers. The presheaf $\tilde{\Gamma}_*(M)$ of abelian groups, defined by*

$$\tilde{\Gamma}_*(M)(X) = A^0(X, M)$$

for any smooth scheme X/k , is a MW-homotopy sheaf canonically isomorphic to M as presheaves.

In Section 4, we recall the construction of the Bass-Tate transfer maps on a contracted homotopy sheaf M_{-1} and prove that this defines a structure of generalized transfers:

Theorem 2 (see **Theorem 4.1.21**). *Let $M \in \mathbf{HI}(k)$ be a homotopy sheaf. Then:*

1. *Assume that 2 is invertible. The rational contracted homotopy sheaf $M_{-1, \mathbb{Q}}$ is a homotopy sheaf with generalized transfers.*
2. *Assuming Conjecture 4.1.13, the contracted homotopy sheaf M_{-1} is a homotopy sheaf with generalized transfers.*

In particular, we obtain the following intersection multiplicity formula which was left open in Feld (2020a):

Theorem 3 (see **Theorem 4.1.16**). *Let $M \in \mathbf{HI}(k)$ be a homotopy sheaf. Consider morphisms of fields over k , $\varphi : E \rightarrow F$ and $\psi : E \rightarrow L$ with φ finite. Let R be the ring $F \otimes_E L$. For each $\mathfrak{p} \in \text{Spec } R$, let $\varphi_{\mathfrak{p}} : L \rightarrow R/\mathfrak{p}$ and $\psi_{\mathfrak{p}} : F \rightarrow R/\mathfrak{p}$ be the morphisms induced by φ and ψ . One has*

$$M_{-1}(\psi) \circ \text{Tr}_{\varphi} = \sum_{\mathfrak{p} \in \text{Spec } R} e_{\mathfrak{p}, \varepsilon} \text{Tr}_{\varphi_{\mathfrak{p}}} \circ M_{-1}(\psi_{\mathfrak{p}})$$

where $e_{\mathfrak{p}, \varepsilon} = \sum_{i=1}^{e_{\mathfrak{p}}} \langle -1 \rangle^{i-1}$ is the quadratic form associated to the length $e_{\mathfrak{p}}$ of the localized ring $R_{(\mathfrak{p})}$.

Calmès and Fasel, generalizing ideas of Voevodsky, introduced the additive symmetric monoidal category $\widetilde{\text{Cor}}_k$ of smooth k -schemes with morphisms given by the so-called *finite Milnor-Witt correspondences* (see (Bachmann et al., 2020, Chapter 2)). In Section 5, we recall the basic definitions regarding this theory and prove that any homotopy sheaf with MW-transfers has a structure of a sheaf with generalized transfers. More precisely, we show that the two notions coincide:

Theorem 4 (Theorem 5.2.5). *There is a pair of functors*

$$\mathbf{HI}^{\text{MW}}(k) \begin{array}{c} \xrightarrow{\tilde{\Gamma}^*} \\ \xleftarrow{\tilde{\Gamma}_*} \end{array} \mathbf{HI}^{\text{gtr}}(k)$$

that forms an equivalence between the category of homotopy sheaves with MW-transfers and the category of homotopy sheaves with generalized transfers.

In Section 6, we prove the following theorem that characterizes the essential image of the functor $\tilde{\gamma}_* : \mathbf{HI}^{\text{MW}}(k) \rightarrow \mathbf{HI}(k)$ that forgets MW-transfers.

Theorem 5 (Theorem 6.1.6). *Let $M \in \mathbf{HI}(k)$ be a homotopy sheaf. The following assertions are equivalent:*

- (i) *There exists $M' \in \mathbf{HI}(k)$ satisfying Conjecture 4.1.13 and such that $M \simeq M'_{-1}$.*
- (ii) *There exists a structure of generalized transfers on M .*
- (iii) *There exists a structure of MW-transfers on M .*
- (iv) *There exists $M'' \in \mathbf{HI}(k)$ such that $M \simeq M''_{-2}$.*

This result is linked with the conservativity conjecture from Bachmann and Yakerson (2020) and allows us to prove the following theorems.

Theorem 6 (Corollary 6.2.3). *Let $d > 0$ be a natural number. The Bachmann-Yakerson conjecture holds (integrally) for $d = 2$ and rationally for $d = 1$: namely, the canonical functor*

$$\mathbf{SH}^{S^1}(k)(2) \rightarrow \mathbf{SH}(k)$$

is conservative on bounded below objects¹, the canonical functor

$$\mathbf{SH}^{S^1}(k)(1) \rightarrow \mathbf{SH}(k)$$

is conservative on rational bounded below objects, and the canonical functor

¹Also known as *connective* objects.

$$\mathbf{HI}(k, \mathbb{Q})(1) \rightarrow \mathbf{HI}^{\mathrm{fr}}(k, \mathbb{Q})$$

is an equivalence of abelian categories.

Moreover, let \mathcal{X} be a pointed motivic space. Then the canonical map

$$\pi_0 \Omega_{\mathbb{P}^1}^d \Sigma_{\mathbb{P}^1}^d \mathcal{X} \rightarrow \pi_0 \Omega_{\mathbb{P}^1}^{d+1} \Sigma_{\mathbb{P}^1}^{d+1} \mathcal{X}$$

is an isomorphism for $d \geq 2$.

In conclusion, we note that the several notions generalizing Voevodsky's theory of homotopy sheaves with transfers are all equivalent:

Theorem 7 (Corollary 6.2.4). *The category of homotopy sheaves with generalized transfers, the category of MW-homotopy sheaves and the category of homotopy sheaves with framed transfers are equivalent:*

$$\mathbf{HI}^{\mathrm{gtr}}(k) \simeq \mathbf{HI}^{\mathrm{MW}}(k) \simeq \mathbf{HI}^{\mathrm{fr}}(k).$$

In future work, we will apply the conservative conjecture of Bachmann and Yakerson to study some intersection points between \mathbb{A}^1 -homotopy theory and affine algebraic geometry. For instance, following (Asok and Østvær, 2019, Conjecture 5.3.11 and Remark 5.3.12), one should obtain:

Theorem 8. *Let X be a smooth scheme and $x \in X$ a closed point. If $\Sigma_{\mathbb{P}^1}^\infty(X, x) \simeq *$ in $\mathbf{SH}(k)$, then $\Sigma_{\mathbb{P}^1}^2(X, x)$ is \mathbb{A}^1 -contractible.*

In particular, this applies when X is a Koras-Russel threefold of the first or second kind (see (Asok and Østvær, 2019, Theorem 5.3.9) and Dubouloz and Fasel (2018); Hoyois et al. (2015) for similar results).

1.2. Outline of the paper

In Section 2, we recall the theory of unramified sheaves and how they are related to homotopy sheaves of abelian groups.

In Section 3, we define the notion of sheaves with generalized transfers and study the associated Rost-Schmid complex.

In Section 4, we define the Bass-Tate transfer maps on a contracted homotopy sheaf M_{-1} and prove the conjecture of Morel in the case of rational coefficients.

In Section 5, we recall the theory of sheaves with MW-transfers Bachmann et al. (2020) and prove that it is equivalent to the notion of sheaves with generalized transfers.

In Section 6, we give some corollaries of Theorem 5.2.5. In particular, we characterize the essential image of the functor $\tilde{\gamma}_* : \mathbf{HI}^{\text{MW}}(k) \rightarrow \mathbf{HI}(k)$ that forgets MW-transfers and use the previous results to discuss the *conservativity conjecture* in \mathbb{A}^1 -homotopy due to Bachmann and Yakerson (see (Bachmann and Yakerson, 2020, Conjecture 1.1) and Bachmann (2020)).

Notation

Throughout the paper, we fix a (commutative) field k and we assume moreover that k is infinite perfect of characteristic not 2. We need these assumptions in order to apply the cancellation theorem (Bachmann et al., 2020, Chapter 4) but we believe these restrictions could be lifted.

We denote by **Grp** and **Ab** the categories of (abelian) groups.

We consider only schemes that are essentially of finite type over k . All schemes and morphisms of schemes are defined over k . The category of smooth k -schemes of finite type is denoted by Sm_k and is endowed with the Nisnevich topology (thus, *sheaf* always means *sheaf for the Nisnevich topology*).

Let X be a scheme and x a point of X . We define the codimension of x in X to be $\dim(\mathcal{O}_{X,x})$, the dimension of the localisation ring of x in X (see also (Stacks Project Authors, 2018, TAG 02IZ)). If n a natural number, we denote by $X_{(n)}$ (resp. $X^{(n)}$) the set of point of dimension n (resp. codimension n) of X (this makes sense even if X is not smooth).

By a field E over k , we mean a *k-finitely generated field* E . Since k is perfect, notice that $\text{Spec } E$ is essentially smooth over S . We denote by \mathcal{F}_k the category of such fields.

Let $f : X \rightarrow Y$ be a (quasi)projective lci morphism of schemes (e.g. a morphism between smooth schemes). Denote by \mathcal{L}_f (or $\mathcal{L}_{X/Y}$) the virtual vector bundle over Y associated with the cotangent complex of f defined as follows: if $p : X \rightarrow Y$ is a smooth morphism, then \mathcal{L}_p is (isomorphic to) $\Omega_{X/Y}$ the space of (Kähler) differentials. If $i : Z \rightarrow X$ is a regular closed immersion, then \mathcal{L}_i is the normal cone $-\mathcal{N}_Z X$. If f is the composite $Y \xrightarrow{i} \mathbb{P}_X^n \xrightarrow{p} X$ with p and i as previously (in other words, if f is lci projective), then \mathcal{L}_f is isomorphic to the virtual tangent bundle $i^* \Omega_{\mathbb{P}_X^n/X} - \mathcal{N}_Y(\mathbb{P}_X^n)$ (see also (Feld, 2020a, Section 9)). Denote by ω_f (or $\omega_{X/Y}$) the determinant of \mathcal{L}_f .

Let X be a scheme and $x \in X$ a point, we denote by $\mathcal{L}_x = (\mathfrak{m}_x/\mathfrak{m}_x^2)^\vee$ and $\omega_{x/X} = \omega_x$ its determinant. Similarly, let v a discrete valuation on a field, we denote by ω_v the line bundle $(\mathfrak{m}_v/\mathfrak{m}_v^2)^\vee$.

Let E be a field (over k) and v a valuation on E . We will always assume that v is discrete. We denote by \mathcal{O}_v its valuation ring, by \mathfrak{m}_v its maximal ideal and by $\kappa(v)$ its residue class field. We consider only valuations of geometric type, that is we assume: $k \subset \mathcal{O}_v$, the residue field $\kappa(v)$ is finitely generated over k and satisfies $\text{tr. deg}_k(\kappa(v)) + 1 = \text{tr. deg}_k(E)$.

Let E be a field. We denote by $\text{GW}(E)$ the Grothendieck-Witt ring of symmetric bilinear forms on E . For any $a \in E^*$, we denote by $\langle a \rangle$ the class of the symmetric bilinear form on E defined by $(X, Y) \mapsto aXY$ and, for any natural number n , we put $n_\varepsilon = \sum_{i=1}^n \langle -1 \rangle^{i-1}$. Recall that, if n and m are two natural numbers, then $(nm)_\varepsilon = n_\varepsilon m_\varepsilon$.

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2. Homotopy sheaves

2.1. Unramified sheaves

In this subsection, we summarize (Morel, 2012, Chapter 2) and recall the basic results concerning unramified sheaves.

Definition 2.1.1. 1. A sheaf of sets \mathcal{S} on Sm_k is said to be \mathbb{A}^1 -invariant if for any $X \in \text{Sm}_k$, the map

$$\mathcal{S}(X) \rightarrow \mathcal{S}(\mathbb{A}_X^1)$$

induced by the projection $\mathbb{A}^1 \times X \rightarrow X$, is a bijection.

2. A sheaf of groups \mathcal{G} on Sm_k is said to be *strongly* \mathbb{A}^1 -invariant if, for any $X \in \text{Sm}_k$, the map

$$H_{\text{Nis}}^i(X, \mathcal{G}) \rightarrow H_{\text{Nis}}^i(\mathbb{A}^1 \times X, \mathcal{G})$$

induced by the projection $\mathbb{A}^1 \times X \rightarrow X$, is a bijection for $i \in \{0, 1\}$.

3. A sheaf of abelian groups M on Sm_k is said to be *strictly \mathbb{A}^1 -invariant* if, for any $X \in \mathrm{Sm}_k$, the map

$$H_{Nis}^i(X, M) \rightarrow H_{Nis}^i(\mathbb{A}^1 \times X, M)$$

induced by the projection $\mathbb{A}^1 \times X \rightarrow X$, is a bijection for $i \in \mathbb{N}$.

Remark 2.1.2. In the sequel, we work with M a sheaf of groups. We could give more general definitions for sheaves of sets but, in practice, we need only the case of sheaves of abelian groups. In that case, we recall that a strongly \mathbb{A}^1 -invariant sheaf of abelian groups is necessarily strictly \mathbb{A}^1 -invariant (see (Morel, 2012, Corollary 5.46)).

Definition 2.1.3. An unramified presheaf of groups M on Sm_k is a presheaf of groups M such that the following holds:

- (0) For any smooth scheme $X \in \mathrm{Sm}_k$ with irreducible components X_α ($\alpha \in X^{(0)}$), the canonical map $M(X) \rightarrow \prod_{\alpha \in X^{(0)}} M(X_\alpha)$ is an isomorphism.
- (1) For any smooth scheme $X \in \mathrm{Sm}_k$ and any open subscheme $U \subset X$ everywhere dense in X , the restriction map $M(X) \rightarrow M(U)$ is injective.
- (2) For any smooth scheme $X \in \mathrm{Sm}_k$, irreducible with function field F , the injective map

$$M(X) \rightarrow \bigcap_{x \in X^{(1)}} M(\mathcal{O}_{X,x})$$

is an isomorphism (the intersection being computed in $M(F)$).

Example 2.1.4. Homotopy modules with transfers Déglise (2011) and Rost cycle modules Rost (1996) define unramified sheaves. In characteristic not 2, the sheaf associated to the presheaf of Witt groups $X \rightarrow W(X)$ is unramified.

We may give an explicit description of unramified sheaves on Sm_k in terms of their sections on fields $F \in \mathcal{F}_k$ and some extra structure. We will say that a functor $M : \mathcal{F}_k \rightarrow \mathbf{Grp}$ is *continuous* if $M(F)$ is the filtering colimit of the groups $M(F_\alpha)$ where F_α runs over the subfields of F of finite type over k .

Definition 2.1.5 (Morel (2012), Definition 2.6). An unramified \mathcal{F}_k -datum consists of:

uD1 A continuous functor $M : \mathcal{F}_k \rightarrow \mathbf{Grp}$.

uD2 For any field $F \in \mathcal{F}_k$ and any discrete valuation v on F , a subgroup

$$M(\mathcal{O}_v) \subset M(F).$$

uD3 For any field $F \in \mathcal{F}_k$ and any valuation v on F , a map

$$s_v : M(\mathcal{O}_v) \rightarrow M(\kappa(v)),$$

called the specialization map associated to v .

The previous data should satisfy the following axioms:

uA1 If $\iota : E \subset F$ is a separable extension in \mathcal{F}_k and w is a valuation on F which restrict to a discrete valuation v on E with ramification index 1, then the arrow $M(\iota)$ maps $M(\mathcal{O}_v)$ into $M(\mathcal{O}_w)$. Moreover, if the induced extension $\bar{\iota} : \kappa(v) \rightarrow \kappa(w)$ is an isomorphism, then the following square

$$\begin{array}{ccc} M(\mathcal{O}_v) & \longrightarrow & M(\mathcal{O}_w) \\ \downarrow & & \downarrow \\ M(E) & \longrightarrow & M(F) \end{array}$$

is cartesian.

uA2 Let $X \in \mathbf{Sm}_k$ be an irreducible smooth scheme with function field F . If $x \in M(F)$, then x lies in all but a finite number of $M(\mathcal{O}_x)$ where x runs over the set $X^{(1)}$ of points of codimension 1.

uA3(i) If $\iota : E \subset F$ is an extension in \mathcal{F}_k and w is a discrete valuation on F which restricts to a discrete valuation v on E , then $M(\iota)$ maps $M(\mathcal{O}_v)$ into $M(\mathcal{O}_w)$ and the following diagram

$$\begin{array}{ccc} M(\mathcal{O}_v) & \longrightarrow & M(\mathcal{O}_w) \\ \downarrow & & \downarrow \\ M(\kappa(v)) & \longrightarrow & M(\kappa(w)) \end{array}$$

is commutative.

uA3(ii) If $\iota : E \subset F$ is an extension in \mathcal{F}_k and w a discrete valuation on F which restricts to zero on E , then the map

$$M(\iota) : M(E) \rightarrow M(F)$$

has its image contained in $M(\mathcal{O}_v)$. Moreover, if $\bar{\iota} : E \subset \kappa(w)$ denotes the induced extension, the composition

$$M(E) \longrightarrow M(\mathcal{O}_v) \xrightarrow{s_v} M(\kappa(w))$$

is equal to $M(\bar{\iota})$.

uA4(i) For any smooth scheme $X \in \text{Sm}_k$ local of dimension 2 with closed point $z \in X^{(2)}$, and for any point $y_0 \in X^{(1)}$ such that the reduced closed scheme \bar{y}_0 is k -smooth, then

$$s_{y_0} : M(\mathcal{O}_{y_0}) \rightarrow M(\kappa(y_0))$$

maps $\bigcap_{y \in X^{(1)}} M(\mathcal{O}_y)$ into $M(\mathcal{O}_{\bar{y}_0, z}) \subset M(\kappa(y_0))$.

uA4(ii) The composition

$$\bigcap_{y \in X^{(1)}} M(\mathcal{O}_y) \rightarrow M(\mathcal{O}_{\bar{y}_0, z}) \rightarrow M(\kappa(z))$$

does not depend on the choice of y_0 such that $\bar{y}_0 \in \text{Sm}_k$.

Example 2.1.6. For any integer n , the functor $\mathbf{K}^{\text{MW}} : \mathcal{F}_k \rightarrow \mathbf{Grp}$ of Milnor-Witt K-theory (defined in (Morel, 2012, Chapter 3) and (Feld, 2020a, Section 1)) is an unramified \mathcal{F}_k -datum.

2.1.7. An unramified sheaf M defines in an obvious way an unramified \mathcal{F}_k -datum. Indeed, taking the evaluation² on the field extensions of k yields a restriction functor:

$$M : \mathcal{F}_k \rightarrow \mathbf{Grp}, F \mapsto M(F)$$

²See the proof of (Morel, 2012, Proposition 2.8) for more details.

such that, for any field F with valuation v , we have an $M(\mathcal{O}_v) \subset M(F)$ and a specialization map $s_v : M(\mathcal{O}_v) \rightarrow M(\kappa(v))$ (obtained by choosing smooth models over k for the closed immersion $\mathrm{Spec} \kappa(v) \rightarrow \mathrm{Spec} \mathcal{O}_v$). We claim that this satisfies axioms uA1, \dots , uA4(ii).

Reciprocally, given an unramified \mathcal{F}_k -datum M and $X \in \mathrm{Sm}_k$ an irreducible smooth scheme with function field F , we define the subset $M(X) \subset M(F)$ as the intersection $\bigcap_{x \in X^{(1)}} M(\mathcal{O}_x) \subset M(F)$. We extend the definition for any X so that property (0) is satisfied. Using the fact that any map $f : Y \rightarrow X$ between smooth schemes is the composition

$$Y \hookrightarrow Y \times_k X \twoheadrightarrow X$$

of closed immersion followed by a smooth projection, one can define an unramified sheaf $M : \mathrm{Sm}_k \rightarrow \mathbf{Grp}$. In short, we have the following theorem.

Theorem 2.1.8 (Morel (2012), Theorem 2.11). *The two functors described above define an equivalence between the category of unramified sheaves on Sm_k and that of unramified \mathcal{F}_k -data.*

Example 2.1.9. Combining Example 2.1.6 and the previous theorem 2.1.8, we obtain a definition of the (unramified) sheaf of Milnor-Witt K-theory $\mathbf{K}_n^{\mathrm{MW}}$ for any integer n .

2.1.10. From now on, we will not distinguish between the notion of unramified sheaves on Sm_k and that of unramified \mathcal{F}_k -datum. In the remaining subsection, we fix M an unramified sheaf of groups on Sm_k and explain how it is related to strongly \mathbb{A}^1 -invariant sheaves.

2.1.11. NOTATION. If $\varphi : E \rightarrow F$ is an extension of fields, the map from uD1

$$M(\varphi) : M(E) \rightarrow M(F)$$

is also denoted by res_φ , $\mathrm{res}_{F/E}$ or φ_* .

2.1.12. Let $F \in \mathcal{F}_k$ be a field and v a valuation on F . We define the pointed set

$$H_v^1(\mathcal{O}_v, M) = M(F)/M(\mathcal{O}_v).$$

This is a left $M(F)$ -set. Moreover, for any point y of codimension 1 in $X \in \text{Sm}_k$, we set $H_y^1(X, M) = H_y^1(\mathcal{O}_{X,y}, M)$. By axiom uA2, if X is irreducible with function field F , the induced left action of $M(F)$ on $\prod_{y \in X^{(1)}} H_y^1(X, M)$ preserves the weak-product

$$\prod'_{y \in X^{(1)}} H_y^1(X, M) \subset \prod_{y \in X^{(1)}} H_y^1(X, M)$$

where the weak-product $\prod'_{y \in X^{(1)}} H_y^1(X, M)$ means the set of families for which all but a finite number of terms are the base point of $H_y^1(X, M)$. By definition and axiom uA2, the isotropy subgroup of this action of $M(F)$ on the base point of $\prod'_{y \in X^{(1)}} H_y^1(X, M)$ is exactly $M(X) = \cap_{y \in X^{(1)}} M(\mathcal{O}_{X,y})$. We summarize this property by saying that the diagram

$$1 \rightarrow M(X) \rightarrow M(F) \Rightarrow \prod'_{y \in X^{(1)}} H_y^1(X, M)$$

is exact.

Definition 2.1.13. For any point z of codimension 2 in a smooth scheme X , we denote by $H_z^2(X, M)$ the orbit set of $\prod'_{y \in X_{(z)}^{(1)}} H_y^1(X, M)$ under the left action of $M(F)$ where $F \in \mathcal{F}_k$ is the function field of $X_{(z)}$.

2.1.14. For an irreducible essentially smooth scheme X with function field F , we define the boundary $M(F)$ -equivariant map

$$\prod'_{y \in X^{(1)}} H_y^1(X, M) \rightarrow \prod_{z \in X^{(2)}} H_z^2(X, M)$$

by collecting together the compositions

$$\prod'_{y \in X^{(1)}} H_y^1(X, M) \rightarrow \prod'_{y \in X_{(z)}^{(1)}} H_y^1(X, M) \rightarrow H_z^2(X, M)$$

for each $z \in X^{(2)}$.

It is not clear in general whether or not the image of the boundary map is always contained in the weak product $\prod'_{z \in X^{(2)}} H_z^2(X, M)$. For this reason Morel introduces the following axiom:

uA2' For any irreducible essentially smooth scheme X , the image of the boundary map

$$\prod'_{y \in X^{(1)}} H_y^1(X, M) \rightarrow \prod_{z \in X^{(2)}} H_z^2(X, M)$$

is contained in the weak product $\prod'_{z \in X^{(2)}} H_z^2(X, M)$.

2.1.15. From now on we assume that M satisfies uA2'. For any smooth scheme X irreducible with function field F , we have a complex $C^*(X, M)$

$$1 \rightarrow M(X) \rightarrow M^{(0)}(X) \Rightarrow M^{(1)}(X) \rightarrow M^{(2)}(X)$$

where

$$M^{(0)}(X) = \prod'_{x \in X^{(0)}} M(\kappa(x)) = \prod_{x \in X^{(0)}} M(\kappa(x)),$$

$$M^{(1)}(X) = \prod'_{y \in X^{(1)}} H_y^1(X, M)$$

and

$$M^{(2)}(X) = \prod'_{z \in X^{(2)}} H_z^2(X, M).$$

By construction, this complex is exact (in an obvious sense, see (Morel, 2012, Definition 2.20)) for any (essentially) smooth local scheme of dimension ≤ 2 .

Definition 2.1.16. A strongly unramified \mathcal{F}_k -data is an unramified \mathcal{F}_k -data M satisfying uA2' and the following axioms:

uA5(i) For any separable finite extension $\iota : E \subset F$ in \mathcal{F}_k , any discrete valuation w on F which restricts to a discrete valuation v on E with ramification index 1, and such that the induced extension $\bar{\iota} : \kappa(v) \rightarrow \kappa(w)$ is an isomorphism, the commutative square of groups

$$\begin{array}{ccc} M(\mathcal{O}_v) & \longrightarrow & M(E) \\ \downarrow & & \downarrow \\ M(\mathcal{O}_w) & \longrightarrow & M(F) \end{array}$$

induces a bijection $H_w^1(\mathcal{O}_w, M) \simeq H_v^1(\mathcal{O}_v, M)$.

uA5(ii) For any étale morphism $X' \rightarrow X$ between smooth local k -schemes of dimension 2, with closed point respectively z' and z , inducing an isomorphism on the residue fields $\kappa(z) \simeq \kappa(z')$, the pointed map

$$H_z^2(X, M) \rightarrow H_{z'}^2(X', M)$$

has trivial kernel.

uA6 For any localization U of a smooth k -scheme at some point u of codimension ≤ 1 , the complex:

$$1 \rightarrow M(\mathbb{A}_U^1) \rightarrow M^{(0)}(\mathbb{A}_U^1) \Rightarrow M^{(1)}(\mathbb{A}_U^1) \rightarrow M^{(2)}(\mathbb{A}_U^1)$$

is exact. Moreover, the morphism $M(U) \rightarrow M(\mathbb{A}_U^1)$ is an isomorphism.

Theorem 2.1.17 (Morel (2012), Theorem 2.27). *There is an equivalence between the category of strongly \mathbb{A}^1 -invariant sheaves of groups on Sm_k and that of strongly unramified \mathcal{F}_k -data of groups on Sm_k .*

Definition 2.1.18. A strongly \mathbb{A}^1 -invariant Nisnevich sheaf of abelian groups is called a *homotopy sheaf*. We denote by $\mathbf{HI}(k)$ the category of homotopy sheaves and natural transformations of sheaves.

Example 2.1.19. As a corollary of the previous theorem, we obtain that the sheaf of Milnor-Witt K-theory $\mathbf{K}_n^{\mathrm{MW}}$ is a homotopy sheaf for any integer n .

2.1.20. MONOIDAL STRUCTURE. Recall that there is a canonical adjunction of categories

$$D(\mathrm{Sh}(\mathrm{Sm}_k)) \xrightleftharpoons[\mathcal{O}]{\pi_{\mathbb{A}^1}} \mathbf{D}_{\mathbb{A}^1}^{\mathrm{eff}}(k)$$

where $\mathbf{D}_{\mathbb{A}^1}^{\mathrm{eff}}(k)$ the effective \mathbb{A}^1 -derived category and $D(\mathrm{Sh}(\mathrm{Sm}_k))$ is the derived category of complexes of sheaves over Sm_k (see (Cisinski and Déglise, 2019, §5)). Thanks to Morel's \mathbb{A}^1 -localization theorem, we can prove that there is a unique t-structure on $\mathbf{D}_{\mathbb{A}^1}^{\mathrm{eff}}(k)$ such that the forgetful functor \mathcal{O} is t-exact and that the category of homotopy sheaves $\mathbf{HI}(k)$ is equivalent to the heart $(\mathbf{D}_{\mathbb{A}^1}^{\mathrm{eff}}(k))^{\heartsuit}$ for this t-structure (in particular, $\mathbf{HI}(k)$ is a Grothendieck category). Since the canonical tensor product $\otimes_{\mathbf{D}_{\mathbb{A}^1}^{\mathrm{eff}}}$ is right t-exact, it induces a monoidal structure on $\mathbf{HI}(k)$. Precisely, if $F, G \in \mathbf{HI}(k)$ are two homotopy sheaves, then their tensor product is

$$F \otimes_{\mathbf{HI}} G = H_0^{\mathbb{A}^1}(F \otimes_{\mathbf{D}_{\mathbb{A}^1}^{\mathrm{eff}}} G)$$

where $H_0^{\mathbb{A}^1}$ is the homology object in degree 0 for the homotopy t-structure.

Example 2.1.21. For any integers n and m , we have a canonical morphism

$$\mathbf{K}_n^{\mathrm{MW}} \otimes_{\mathbf{HI}} \mathbf{K}_m^{\mathrm{MW}} \rightarrow \mathbf{K}_{n+m}^{\mathrm{MW}}.$$

In particular $\mathrm{GW} = \mathbf{K}_0^{\mathrm{MW}}$ is a commutative monoid.

2.2. Contracted homotopy sheaves

2.2.1. In this section, we fix $M \in \mathbf{HI}(k)$ a homotopy sheaf. Recall that the contraction M_{-1} is by definition the sheaf of abelian groups

$$X \mapsto \ker(M(\mathbb{G}_m \times X) \rightarrow M(X)).$$

According to (Morel, 2012, Lemma 2.32), the sheaf M_{-1} is also a homotopy sheaf and is called a *contracted homotopy sheaf*. Morel also proved that we have

$$M_{-1} = \underline{\mathcal{H}om}(\mathbf{K}_1^{\text{MW}}, M)$$

where $\underline{\mathcal{H}om}$ is the internal hom-object of the abelian category of sheaves of abelian groups over Sm_k .

Example 2.2.2. For any integer n , we have a canonical isomorphism

$$(\mathbf{K}_n^{\text{MW}})_{-1} = \mathbf{K}_{n-1}^{\text{MW}}$$

according to (Morel, 2012, Corollary 6.43).

2.2.3. For any smooth scheme X , we have a short exact sequence

$$0 \rightarrow M(X) \rightarrow M(\mathbb{G}_m \times X) \rightarrow M_{-1}(X) \rightarrow 0.$$

Following (Morel, 2012, §3.3), we let $\mathcal{O}(X)^\times$ act on $M(\mathbb{G}_m \times X)$ by translation through the map $(u, x) \mapsto U^*(x)$ where $U : \mathbb{G}_m \times X \simeq \mathbb{G}_m \times X$ is the automorphism multiplication by the unit $u \in \mathcal{O}(X)^\times$. If we let $\mathcal{O}(X)^\times$ act trivially on $M(X)$, then the above left inclusion is equivariant and thus M_{-1} gets a canonical and functorial structure of \mathbb{G}_m -module.

According (Morel, 2012, Lemma 3.49), the \mathbb{G}_m -module structure on M_{-1} is induced from a $(\text{GW} = \mathbf{K}_0^{\text{MW}})$ -module structure on M_{-1} through the morphism of sheaves $\mathbb{G}_m \rightarrow \mathbf{K}_0^{\text{MW}}$ that maps a unit u to its symbol $\langle u \rangle = 1 + \boldsymbol{\eta}[u]$.

Moreover, we have a bilinear pairing

$$\begin{aligned} \mathbf{K}_1^{\text{MW}} \times M_{-1} &\rightarrow M \\ ([u], \mu) &\mapsto [u] \cdot \mu, \end{aligned}$$

where \mathbf{K}_1^{MW} is defined in 2.1.19 (see also (Morel, 2012, Lemma 3.48)).

2.2.4. Let $N \in \mathbf{HI}(k)$ be another homotopy sheaf and assume N is equipped with a GW-module structure. Let $X \in \mathbf{Sm}_k$ be a smooth scheme and \mathfrak{L} a line bundle on X . We define the twist of N by \mathfrak{L} denoted by $N\{\mathfrak{L}\}$ or $N \otimes_{\mathbb{Z}[\mathbf{G}_m]} \mathbb{Z}[\mathfrak{L}^\times]$ as the sheaf associated to the presheaf on the Zariski site X_{Zar} :

$$U \mapsto N(U) \otimes_{\mathbb{Z}[\mathcal{O}_X(U)^\times]} \mathbb{Z}[\mathfrak{L}_U^\times]$$

where \mathfrak{L}_U^\times is the set of isomorphisms between \mathcal{O}_U and \mathfrak{L}_U (which may be empty). We put $N(X, \mathfrak{L}) = \Gamma(X, N\{\mathfrak{L}\})$. We extend the definition to any essentially smooth scheme X/k . If $X = \text{Spec}(F)$ is the spectrum of a field, then the line bundle \mathfrak{L} corresponds to an invertible F -vector space, and we have $N(X, \mathfrak{L}) = N(X) \otimes_{\mathbb{Z}[\mathcal{O}_X(X)^\times]} \mathbb{Z}[\mathfrak{L}_X^\times]$. In particular, this definition applies to the sheaf $N = M_{-1}$.

2.2.5. COHOMOLOGY WITH SUPPORT EXACT SEQUENCE. Let $M \in \mathbf{HI}(k)$ be a homotopy sheaf. For any closed immersion $i : Z \rightarrow X$ of smooth schemes over k , with complementary open immersion $j : U \rightarrow X$, there exists a canonical cohomology with support exact sequence of the form:

$$\Gamma_Z(X, M) \xrightarrow{i_*} M(X) \xrightarrow{j^*} M(U) \xrightarrow{\partial} H_Z^1(X, M) \longrightarrow \dots$$

2.2.6. Recall that, by convention, we implicitly extend our sheaves to the category of essentially smooth schemes in a canonical way. The purity isomorphism (more precisely: Axiom uA5(i) and (Morel, 2012, Lemma 3.50)) implies that for any discrete valuation v on a field $F \in \mathcal{F}_k$, one has a canonical bijection

$$H_v^1(\mathcal{O}_v, M) \simeq M_{-1}(\kappa(v), \omega_v)$$

and thus obtain a residue map

$$\partial_v : M(F) \rightarrow M_{-1}(\kappa(v), \omega_v).$$

Proposition 2.2.7. *Let $M \in \mathbf{HI}(k)$ be a homotopy sheaf and consider the following commutative square*

$$\begin{array}{ccc} T & \xhookrightarrow{k} & Y \\ q \downarrow & & \downarrow p \\ Z & \xhookrightarrow{i} & X \end{array}$$

of closed immersions of separated schemes over k . We have the following diagram

$$\begin{array}{ccccccc}
\Gamma_T(X, M) & \xrightarrow{k_*} & \Gamma_Y(X, M) & \xrightarrow{k'^*} & \Gamma_{Y-T}(X-Z, M) & \xrightarrow{\partial_k} & H_T^1(X, M) \\
q_* \downarrow & & \downarrow p_* & & \downarrow \tilde{p}_* & & \downarrow q_* \\
\Gamma_Z(X, M) & \xrightarrow{i_*} & M(X) & \xrightarrow{i'^*} & M(X-Z) & \xrightarrow{\partial_i} & H_Z^1(X, M) \\
q'^* \downarrow & & \downarrow p'^* & & \downarrow \tilde{p}'^* & & \downarrow q'^* \\
\Gamma_{Z-T}(X-Y, M) & \xrightarrow{\tilde{i}_*} & M(X-Y) & \xrightarrow{\tilde{i}'^*} & M(X-(Y \cup Z)) & \xrightarrow{\partial_{\tilde{i}}} & H_{Z-T}^1(X-T, M) \\
\downarrow \partial_k & & \downarrow \partial_p & & \downarrow \partial_{\tilde{p}} & (*) & \downarrow \partial_{\tilde{k}} \\
H_T^1(X, M) & \xrightarrow{k_*} & H_Y^1(X, M) & \xrightarrow{k'^*} & H_{Y-T}^1(X-Z, M) & \xrightarrow{\partial_k} & H_T^2(X, M)
\end{array}$$

with obvious maps. Each squares of this diagram is commutative except for $(*)$ which is anti-commutative.

PROOF. This is a classical consequence of the octahedron axiom.

3. Sheaves with generalized transfers

3.1. Morel's axioms

The following definition is a slightly improved version of Morel's definition in (Morel, 2011, Definition 5.7). The main purpose is to give an axiomatization of the Bass-Tate transfers defined in Section 4. It is directly inspired by Rost's theory of cycles modules. Lastly, it can be seen as an *effective* counter-part of our own axiomatic of MW-cycle modules (see Feld (2020a)).

Definition 3.1.1. Let M be a homotopy sheaf (i.e. a strongly \mathbb{A}^1 -invariant Nisnevich sheaf of abelian groups on Sm_k). We say that M has a structure of generalized transfers if M has a structure of GW-module and satisfies the following datum

eD2 For any finite extension $\varphi : E \rightarrow F$ in \mathcal{F}_k and any natural number $i \in \mathbb{N}$, there is a map

$$\text{Tr}_\varphi = \text{Tr}_{F/E} : M_{-i}(F, \omega_{F/k}) \rightarrow M_{-i}(E, \omega_{E/k})$$

called the transfer morphism from F to E .

In addition, this datum satisfies the following axioms:

eR1b $\text{Tr}_{\text{Id}_E} = \text{Id}_{M(E)}$ and for any composable finite morphisms φ and ψ in \mathcal{F}_k , we have

$$\text{Tr}_{\psi \circ \varphi} = \text{Tr}_\varphi \circ \text{Tr}_\psi.$$

eR1c Consider $\varphi : E \rightarrow F$ and $\psi : E \rightarrow L$ with φ finite. Let R be the ring $F \otimes_E L$. For each $\mathfrak{p} \in \text{Spec } R$, let $\varphi_{\mathfrak{p}} : L \rightarrow R/\mathfrak{p}$ and $\psi_{\mathfrak{p}} : F \rightarrow R/\mathfrak{p}$ be the morphisms induced by φ and ψ . One has

$$M(\psi) \circ \text{Tr}_\varphi = \sum_{\mathfrak{p} \in \text{Spec } R} e_{\mathfrak{p}, \varepsilon} \text{Tr}_{\varphi_{\mathfrak{p}}} \circ M(\psi_{\mathfrak{p}})$$

where $e_{p, \varepsilon} = \sum_{i=1}^{e_p} \langle -1 \rangle^{i-1}$ is the quadratic form associated to the length e_p of the local Artinian ring $R_{(\mathfrak{p})}$.

eR2 Let $\psi : E \rightarrow F$ be a finite extension of fields.

eR2b For $\langle a \rangle \in \text{GW}(E)$ and $\mu \in M(F, \omega_{F/k})$, one has $\text{Tr}_{F/E}(\langle \psi(a) \rangle \cdot \mu) = \langle a \rangle \cdot \text{Tr}_{F/E}(\mu)$.

eR2c For $\langle a \rangle \in \text{GW}(F, \omega_{F/k})$ and $\mu \in M(E)$, one has $\text{Tr}_{F/E}(\langle a \rangle \cdot \text{res}_{F/E}(\mu)) = \text{Tr}_{F/E}(\langle a \rangle) \cdot \mu$.

eR3b Let $i \in \mathbb{N}$ be a natural number, $\varphi : E \rightarrow F$ be a finite extension of fields and let v be a valuation on E . For each extension w of v , we denote by $\varphi_w : \kappa(v) \rightarrow \kappa(w)$ the induced morphism. We have

$$\partial_v \circ \text{Tr}_\varphi = \sum_w \text{Tr}_{\varphi_w} \circ \partial_w$$

where $\partial_v : M_{-i}(E, \omega_{E/k}) \rightarrow M_{-i-1}(\kappa(v), \omega_{\kappa(v)/k})$ and $\partial_w : M_{-i}(F, \omega_{F/k}) \rightarrow M_{-i-1}(\kappa(w), \omega_{\kappa(w)/k})$ are the residue maps defined in 2.2.6.

Remark 3.1.2. Our definition differs from Morel's in two ways. First, we have taken into account the twists naturally arising (this is not really important if one works in characteristic zero). Second, Axiom **A3** of (Morel, 2011, Definition 5.7) is replaced by eR3b (we expect these two axioms to be equivalent in characteristic 0).

Remark 3.1.3. We know from 2.1.17 that homotopy sheaves can be understood as certain functors on function fields. Taking into account this fact, our axioms are indeed *effective* variants of that of Milnor-Witt cycle modules. In fact, we will see that they correspond to homotopy sheaves with Milnor-Witt transfers, as MW-cycle modules correspond to homotopy modules. (This explains our choice of numbering of the axioms.)

Remark 3.1.4. A homotopy sheaf with generalized transfers is a particular case of a sheaf with \mathbb{A}^1 -transfers as defined in (Bachmann and Yakerson, 2020, §5).

Example 3.1.5. 1. The Milnor-Witt sheaf \mathbf{K}_n^{MW} has a structure of generalized transfers ((Morel, 2012, §4.2), (Feld, 2020a, Theorem 4.13), or (Feld, 2020b, Theorem 1); see also Theorem 4.1.16).

2. Let M be a homotopy sheaf with generalized transfers, F/E a finite extension of fields and \mathfrak{L}_E a line vector space over E . Then we put $\mathfrak{L}_F = \mathfrak{L}_E \otimes_E F$ which is F -vector space of rank 1. For any natural number n , then transfer maps $\text{Tr}_{F/E}$ of M define morphisms

$$M_{-i}(F, \omega_{F/k} \otimes_F \mathfrak{L}_F) \rightarrow M_{-i}(E, \omega_{E/k} \otimes_E \mathfrak{L}_E)$$

which satisfies axioms eR1b, ..., eR3b. An abuse of language would be to say that if M has a structure of generalized transfers, then so does $M\{\mathfrak{L}\}$.

3. If M is a homotopy sheaf with generalized transfers, then so is M_{-1} .

Remark 3.1.6. In the next section, we will give conditions for a contracted homotopy sheaf M_{-1} to be equipped with a structure of generalized transfers.

Definition 3.1.7. A map between homotopy sheaves with generalized transfers is a natural transformation commuting with the GW -module structure and the transfers. We denote by $\mathbf{HI}^{\text{gtr}}(k)$ the category of homotopy sheaves with generalized transfers over k .

3.1.8. ROST-SCHMID COMPLEX. Let $M \in \mathbf{HI}^{\text{gtr}}(k)$ and let X be a scheme essentially of finite type over k . We define the Rost-Schmid complex as the graded abelian group defined for any $n \in \mathbb{N}$ by:

$$C^n(X, M) = \bigoplus_{x \in X^{(n)}} M_{-n}(\kappa(x), \omega_{\kappa(x)/X}).$$

The transition maps d are defined as follows. If X is normal with generic point ξ , then for any $x \in X^{(1)}$ the local ring of X at x is a valuation ring so that we have a map $\partial_x : M_{-n}(\kappa(\xi), \omega_{\kappa(\xi)/k}) \rightarrow M_{-n-1}(\kappa(x), \omega_{\kappa(x)/k})$ for any n .

Now suppose X is a scheme essentially of finite type over k and let x, y be two points in X . We define a map

$$\partial_y^x : M_{-n}(\kappa(x), \omega_{\kappa(x)/X}) \rightarrow M_{-n-1}(\kappa(y), \omega_{\kappa(y)/X})$$

as follows. Let $Z = \overline{\{x\}}$. If $y \notin Z^{(1)}$, then put $\partial_y^x = 0$. If $y \in Z^{(1)}$, let $\tilde{Z} \rightarrow Z$ be the normalization and put

$$\partial_y^x = \sum_{z|y} \text{Tr}_{\kappa(z)/\kappa(y)} \circ \partial_z$$

with z running through the finitely many points of \tilde{Z} lying over y .

Thus, we may define a differential map

$$d = \sum_{x,y} \partial_y^x : C^n(X, M) \rightarrow C^{n+1}(X, M)$$

which is well-defined according to the following proposition.

Remark 3.1.9. If $M \in \mathbf{HI}(k)$ is a homotopy sheaf, Morel defined in (Morel, 2012, Chapter 4) a complex (also called *the Rost-Schmid complex*) denoted by $C_{RS}^*(X, M)$ (where X is a smooth scheme). If $M \in \mathbf{HI}^{\text{gtr}}(k)$ has generalized transfers, then we have an isomorphism

$$C^n(X, M) \simeq C_{RS}^n(X, M) \tag{3.1.1}$$

for any smooth scheme X and any $n \in \mathbb{N}$.

For now, the above map 3.1.1 is just an isomorphism of abelian groups; but we will see later that it is moreover compatible with the differentials, i.e. we have an isomorphism of complexes. Indeed, the differentials of $C^*(X, M)$ and $C_{RS}^*(X, M)$ are defined exactly in the same manner except for the fact that Morel uses transfer maps that arise automatically on contractions (see Section 4) while ours are given as extra data. In Theorem 4.2.3, we prove that these two types of transfers are equivalent (in other words, our definition of *generalized transfers* is a good axiomatization of the transfers defined in (Morel, 2012, Chapter 4)).

We now define the analogue of Rost's four basic maps (Rost, 1996, §3) for the Rost-Schmid complex and prove that they are morphisms of quasi-complexes in some special cases.

3.1.10. NOTATION Let M and N be two homotopy sheaves which are also GW-modules, X and Y two smooth schemes, \mathfrak{L}_X and \mathfrak{L}_Y two line bundle over X and Y respectively, $U \subset X$ and $V \subset Y$ two subsets, and a morphism

$$\alpha : \bigoplus_{x \in U} M(\kappa(x), (\mathfrak{L}_X)_x) \rightarrow \bigoplus_{y \in V} N(\kappa(y), (\mathfrak{L}_Y)_y).$$

Then we denote by $\alpha_y^x : M(\kappa(x), (\mathfrak{L}_X)_x) \rightarrow N(\kappa(y), (\mathfrak{L}_Y)_y)$ the components of α .

3.1.11. PULLBACK. Let $M \in \mathbf{HI}^{\text{gtr}}(k)$. Let $f : X \rightarrow Y$ be an essentially smooth morphism of schemes essentially of finite type. Define

$$f^* : C^*(Y, M) \rightarrow C^*(X, M)$$

as follows. If $x \in X$ and $y \in Y$ satisfy $f(x) = y$, then $(f^*)_x^y = \Theta \circ \text{res}_{\kappa(x)/\kappa(y)}$, where Θ is the canonical isomorphism induced by $\omega_{\text{Spec } \kappa(x)/\text{Spec } \kappa(y)} \simeq \omega_{X/Y} \times_X \text{Spec } \kappa(x)$. Otherwise, $(f^*)_x^y = 0$. If X is not connected, take the sum over each connected component.

3.1.12. PUSHFORWARD. Let $M \in \mathbf{HI}^{\text{gtr}}(k)$. Let $f : X \rightarrow Y$ be a morphism between schemes essentially of finite type over k and assume that X is connected (if X is not connected, take the sum over each connected component). Let $d = \dim(Y) - \dim(X)$. We define

$$f_* : C^*(X, M\{\omega_f\}) \rightarrow C^{*-d}(Y, M_{-d})$$

as follows. If $x \in X$ and $y \in Y$ satisfies $y = f(x)$ and $\kappa(x)$ is finite over $\kappa(y)$, then put $(f_*)_x^y = \text{Tr}_{\kappa(x)/\kappa(y)}$ where $\text{Tr}_{\kappa(x)/\kappa(y)}$ is the transfer map eD2. Otherwise, put $(f_*)_x^y = 0$.

3.1.13. GW-ACTION. Let $M \in \mathbf{HI}^{\text{gtr}}(k)$. Let X be a scheme essentially of finite type over k and $a \in \mathcal{O}_X^*$ a global unit. Define a morphism

$$\langle a \rangle : C^*(X, M) \rightarrow C^*(X, M)$$

as follows. Let $x, y \in X^{(p)}$ and $\rho \in M_{-*}(\kappa(x), \omega_{x/k})$. If $x = y$, then $\langle a \rangle_x^y(\rho) = \langle a(x) \rangle \cdot \rho$. Otherwise, $\langle a \rangle_x^y(\rho) = 0$.

3.1.14. BOUNDARY MAPS. Let $M \in \mathbf{HI}^{\text{gtr}}(k)$. Let X be a scheme essentially of finite type over k , let $i : Z \rightarrow X$ be a closed immersion and let $j : U = X \setminus Z \rightarrow X$ be the inclusion of the open complement. We will refer to (Z, i, X, j, U) as a boundary triple and define

$$\partial = \partial_Z^U : C_p(U, M) \rightarrow C_{p-1}(Z, M)$$

by taking ∂_y^x to be as the definition in 3.1.8 with respect to X . The map ∂_Z^U is called the boundary map associated to the boundary triple, or just the boundary map for the closed immersion $i : Z \rightarrow X$.

We now fix $M \in \mathbf{HI}^{\text{gtr}}(k)$ a homotopy sheaf with generalized transfers and study the morphisms defined on the Rost-Schmid complex with coefficients in M .

Proposition 3.1.15 (Functoriality and base change). *1. Let $f : X \rightarrow Y$ and $f' : Y \rightarrow Z$ be two morphisms of schemes essentially of finite type. Then*

$$(f' \circ f)_* = f'_* \circ f_*.$$

2. Let $g : Y \rightarrow X$ and $g' : Z \rightarrow Y$ be two essentially smooth morphisms. Then (up to the canonical isomorphism given by $\omega_{Z/X} \simeq \omega_{Z/Y} + (g')^ \omega_{Y/X}$):*

$$(g \circ g')^* = g'^* \circ g^*.$$

3. Consider a pullback square

$$\begin{array}{ccc} U & \xrightarrow{g'} & Z \\ f' \downarrow & & \downarrow f \\ Y & \xrightarrow{g} & X \end{array}$$

with f, f', g, g' as previously. Then

$$g^* \circ f_* = f'_* \circ g'^*$$

up to the canonical isomorphism induced by $\omega_{U/Z} \simeq \omega_{Y/X} \times_Y U$.

PROOF. This follows as in (Feld, 2020a, Proposition 6.1) from eR1b and eR1c.

Proposition 3.1.16. *(i) Let $f : X \rightarrow Y$ be a finite morphism of schemes essentially of finite type. Then*

$$d_Y \circ f_* = f_* \circ d_X.$$

(ii) Let $g : Y \rightarrow X$ be an essentially smooth morphism. Then

$$g^* \circ d_X = d_Y \circ g^*.$$

(iii) Let a be a unit on X . Then

$$d_X \circ \langle a \rangle = \langle a \rangle \circ d_X$$

(iv) Let (Z, i, X, j, U) be a boundary triple. Then

$$d_Z \circ \partial_Z^U = -\partial_Z^U \circ d_U.$$

PROOF. As in (Feld, 2020a, Proposition 6.6), the assertions (ii), (iii) and (iv) follow easily from the definitions.

The assertion (i) is nontrivial since our axioms are weaker than in the stable case studied in (Feld, 2020a, §6). Fortunately, Morel proved that the map

$$g_* : C_*(Y, N_{-1}) \rightarrow C_*(X, N_{-1})$$

is a morphism of (quasi)complexes when N is a homotopy sheaf (see (Morel, 2012, Corollary 5.30))³. The proof can be adapted almost verbatim if we replace the contracted homotopy sheaf N_{-1} by any sheaf with generalized transfers M (the use of (Morel, 2012, Theorem 5.19) is replaced by eR3b). We give more details below.

First of all, we may reduce (as in (Morel, 2012, Corollary 5.30)) to the case where $X = \operatorname{Spec}(B)$ and $Y = \operatorname{Spec}(A)$, with A and B being essentially k -smooth of dimension 1.

³Beware that some results in (Morel, 2012, Chapter 4) may contain typographical mistakes.

Denote by K and L the fraction fields of A and B , respectively. Let $K \subset E \subset L$ an intermediate (finite) extension of fields and C the integral closure of A in E . By the functorial property eR1b of the generalized transfers, we see that, if the assertion (i) holds for $\mathrm{Spec}(B) \rightarrow \mathrm{Spec}(C)$ and for $\mathrm{Spec}(C) \rightarrow \mathrm{Spec}(A)$, then it holds for the composition $\mathrm{Spec}(B) \rightarrow \mathrm{Spec}(A)$. We emphasize the need of the axiom eR1b which corresponds to Conjecture 4.1.13 in (Morel, 2012, Chapter 4) where Morel works with general homotopy sheaves (without transfers).

We may now conclude as in the proof of (Morel, 2012, Theorem 5.26) but let us explain the proof in characteristic 0. To check the assertion, we may moreover assume that A is an henselian essentially k -smooth d.v.r. (in that case, B is also an henselian essentially k -smooth d.v.r.). According to (Morel, 2012, Remark 5.28), there exists a finite filtration $K \subset L_1 \subset \cdots \subset L_r = L$ such that letting $B_i \subset L_i$ be the integral closure (which is also an henselian d.v.r.) each extension $B_{i-1} \subset B_i$ is monogenous. One can then apply axiom eR3b at each step to prove the result.

Theorem 3.1.17. *For any homotopy sheaf with generalized transfers $M \in \mathbf{HI}^{\mathrm{gtr}}(k)$ and any essentially smooth scheme X , the Rost-Schmid complex $C^*(X, M)$ is a complex.*

PROOF. We follow the proof of Morel (see (Morel, 2012, Theorem 5.31)).

Let $z \in X$ be a point of codimension n and let Y be an integral closed subscheme of codimension $n - 2$ with generic point y . We want to prove that the component of $\partial \circ \partial$ starting from the summand $M_{-n+1}(\kappa(y), \omega_{y/X})$ to $M_{-n-1}(\kappa(z), \omega_{z/X})$ is zero. We may reduce to the case where X is of finite type, affine and smooth over $\kappa(z)$ and z is a closed point of codimension n in X . By the Normalization lemma (Serre, 1965, Théorème 2 p.57), there exists a finite morphism $X \rightarrow \mathbb{A}_{\kappa(z)}^n$ such that z maps to 0 (with same residue field) and such that the image of Y is a linear $\mathbb{A}_{\kappa(z)}^2 \subset \mathbb{A}_{\kappa(z)}^n$. Using the pushforward maps (see Proposition 3.1.16), we may reduce to the case $X = \mathbb{A}_{\kappa(z)}^n$, $Y = \mathbb{A}_{\kappa(z)}^2$ and $z = 0$. The proof of this last particular case is exactly the same as (Morel, 2012, Corollary 5.29).

Definition 3.1.18. For any homotopy sheaf with generalized transfers $M \in \mathbf{HI}^{\mathrm{gtr}}(k)$ and any essentially smooth scheme X , the cohomology groups associated to the Rost-Schmid complex are denoted by $A^*(X, M)$.

Remark 3.1.19. According to Proposition 3.1.16, the morphisms f_* for f finite, g^* for g essentially smooth, multiplication by $\langle a \rangle$ commute with the differentials. We use the same notations to denote the induced morphisms on the cohomology groups $A^*(X, M)$.

3.1.20. PRODUCT By definition, if $M \in \mathbf{HI}^{\text{gtr}}$, then there is a structure of \mathbf{K}_0^{MW} -module on M (since $\text{GW} \simeq \mathbf{K}_0^{\text{MW}}$). As in (Rost, 1996, §14), (Fasel, 2020, §3.4) or (Feld, 2020a, §11), we can define, for any smooth k -schemes Z and Y , and any numbers p, q , a product map

$$\times_\mu : C^p(Y, \mathbf{K}_p^{\text{MW}}) \times C^q(Z, M) \rightarrow C^{p+q}(Y \times Z, M)$$

which induces a map on the cohomology groups:

$$\times_\mu : A^p(Y, \mathbf{K}_p^{\text{MW}}) \times A^q(Z, M) \rightarrow A^{p+q}(Y \times Z, M).$$

3.2. MW-transfers structure

Definition 3.2.1. Let $M \in \mathbf{HI}^{\text{gtr}}(k)$ be a homotopy sheaf with generalized transfers. For any smooth scheme X , denote by

$$\tilde{\Gamma}_*(M)(X) = A^0(X, M).$$

By definition, if X is integral with function field $\kappa(X)$, then

$$A^0(X, M) = \ker d = \bigcap_{x \in X^{(1)}} \ker \partial_x \subset M(\kappa(X)).$$

Thus we see that $\tilde{\Gamma}_*(M)(X)$ is canonically isomorphic to $M(X)$ (since M is unramified, recall Definition 2.1.3, Theorem 2.1.8 and Theorem 2.1.17).

This defines a presheaf $\tilde{\Gamma}_*(M)$ where the pullback morphisms are defined to be the same as M .

Remark 3.2.2. The previous definition (along with Proposition 3.1.16) allows us to consider pushforward maps on $\tilde{\Gamma}_*(M)$ for finite morphisms.

Theorem 3.2.3. Let $M \in \mathbf{HI}^{\text{gtr}}(k)$ be a homotopy sheaf with generalized transfers. Then the contravariant functor $\tilde{\Gamma}_*(M) : X \mapsto \tilde{\Gamma}_*(M)(X)$ induces a presheaf on $\widetilde{\text{Cor}}_k$.

PROOF. Let X, Y be two smooth schemes (which may be assumed to be connected without loss of generality). Let β be an element in $\tilde{\Gamma}_*(M)(Y)$. We use the notations and definitions of Subsection 5.1, and fix α a finite correspondence from X to Y with support in T , with $T \subset X \times Y$ an admissible subset. We set $\alpha \cdot p_Y^*(\beta) = \delta_{X \times Y}^*(\alpha \times_\mu p_Y^*(\beta))$ where \times_μ is the product defined in 3.1.20 and $\delta_{X \times Y} : X \times Y \rightarrow (X \times Y) \times (X \times Y)$ is the diagonal morphism. In order to define Milnor-Witt transfers, we put

$$\alpha^*(\beta) = (p_X)_*(\alpha \cdot p_Y^*(\beta))$$

where $p_X : T \rightarrow X$ is the canonical morphism. We remark that the map $(p_X)_*$ is well-defined⁴ thanks to Proposition 3.1.16(i). This yields to an application α^* which is additive. We can see that this definition does not depend on the choice of T . Thus $\alpha \mapsto \alpha^*$ defines a map $\widetilde{\text{Cor}}_k(X, Y) \rightarrow \text{Hom}_{\mathcal{A}b}(\tilde{\Gamma}_*(M)(Y), \tilde{\Gamma}_*(M)(X))$. It remains to check that this map preserves the respective compositions. Consider the diagram

$$\begin{array}{c}
 X \times Z \xleftarrow{q_{XZ}} X \times Y \times Z \xrightarrow{q_{YZ}} Y \times Z \xrightarrow{q_Z} Z \\
 \quad \quad \quad \downarrow q_{XY} \quad \quad \quad \downarrow p_Y \\
 \quad \quad \quad X \times Y \xrightarrow{q_Y} Y \\
 \quad \quad \quad \downarrow p_X \\
 \quad \quad \quad X
 \end{array}$$

$\xleftarrow{r_Z} \quad \quad \quad \xleftarrow{r_X}$

Let $\alpha_1 \in \widetilde{\text{CH}}_{T_1}^{d_Y}(X \times Y, \omega_Y)$ and $\alpha_2 \in \widetilde{\text{CH}}_{T_2}^{d_Z}(X \times Z, \omega_Y)$ be two correspondences, with $T_1 \subset X \times Y$ and $T_2 \subset Y \times Z$ admissible. Moreover, let $\beta \in M(Z)$. By definition, we have

$$(\alpha_2 \circ \alpha_1)^*(\beta) = (r_X)_*[(q_{XY})_*(p_{XY}^* \alpha_1 \cdot q_{YZ}^* \alpha_2) \cdot r_Z^* \beta].$$

Write temporarily $q = q_{XY}$,

$$\begin{aligned}
 \delta_{XYZ} : (X \times Y \times Z) &\rightarrow (X \times Y \times Z) \times (X \times Y \times Z), \\
 \delta_{XY} : (X \times Y) &\rightarrow (X \times Y) \times (X \times Y)
 \end{aligned}$$

⁴We have used Voevodsky's trick here: $X \times Y \mapsto X$ is not finite, but its restriction $p_X : T \rightarrow X$ is finite by assumption.

the diagonal maps, $x = (p_{XY}^* \alpha_1 \cdot q_{YZ}^* \alpha_2)$ and $y = r_Z^* \beta$. We have the following projection formula:

$$\begin{aligned}
q_*(x) \cdot y &= \delta_{XY}^*(q \times 1)_*(x \times_\mu y) & (1) \\
&= q_*((1 \times q)\delta_{XYZ})^*(x \times_\mu y) & (2) \\
&= q_*\delta_{XYZ}^*(1 \times q)^*(x \times_\mu y) & (3) \\
&= q_*(x \cdot q^*(y)) & (4)
\end{aligned}$$

where the equality (1) follows from the axiom eR2c and the definition of the product, the equality (2) from the base change property 3.1.15 applied to the Cartesian square

$$\begin{array}{ccc}
X \times Y \times Z & \xrightarrow{(1 \times q)\delta_{XYZ}} & (X \times Y \times Z) \times (X \times Y) \\
q \downarrow & & \downarrow q \times 1 \\
X \times Y & \xrightarrow{\delta_{XY}} & (X \times Y) \times (X \times Y),
\end{array}$$

the equality (3) from functoriality and the equality (4) from the compatibility of the pullbacks with the GW -action and the definition of the product.

Using the above projection formula, we have

$$\begin{aligned}
(r_X)_*[(q_{XY})_*(p_{XY}^* \alpha_1 \cdot q_{YZ}^* \alpha_2) \cdot r_Z^* \beta] &= (r_X)_*[(q_{XY})_*(p_{XY}^* \alpha_1 \cdot q_{YZ}^* \alpha_2 \cdot q_{XY}^* r_Z^* \beta)] \\
&= (p_X)_*(p_{XY})_*(p_{XY}^* \alpha_1 \cdot q_{YZ}^* \alpha_2 \cdot q_{XZ}^* r_Z^* \beta).
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\alpha_1^* \circ \alpha_2^*(\beta) &= \alpha_1^*((p_Y)_*(\alpha_2 \cdot q_Z^* \beta)) \\
&= (p_X)_*(\alpha_1 \cdot q_Y^*(p_Y)_*(\alpha_2 \cdot q_Z^* \beta))
\end{aligned}$$

By base change 3.1.15, $q_Y^*(p_Y)_* = (p_{XY})_* q_{YZ}^*$ and it follows (using the projection formula once again) that

$$\begin{aligned}
\alpha_1^* \circ \alpha_2^*(\beta) &= (p_X)_*(\alpha_1 \cdot (p_{XY})_*(q_{YZ}^* \alpha_2 \cdot q_{YZ}^* q_Z^* \beta)) \\
&= (p_X)_*(p_{XY})_*(p_{XY}^* \alpha_1 \cdot q_{YZ}^* \alpha_2 \cdot q_{XZ}^* r_Z^* \beta).
\end{aligned}$$

Hence the result.

We have proved the following theorem.

Theorem 3.2.4. *Let $M \in \mathbf{HI}^{\text{gtr}}(k)$ be a homotopy sheaf with generalized transfers. The presheaf $\tilde{\Gamma}_*(M)$ of abelian groups, defined by*

$$\tilde{\Gamma}_*(M)(X) = A^0(X, M)$$

for any smooth scheme X/k , is a MW-homotopy sheaf (see Definition 5.1.10) canonically isomorphic to M as presheaves.

PROOF. Theorem 3.2.3 implies that $\tilde{\Gamma}_*(M)$ is an MW-homotopy sheaf. The second assertion follows from the fact that the natural map $M \rightarrow \tilde{\Gamma}_*(M)$ of presheaves can be identify with the identity thanks to Definition 3.2.1 and Theorem 2.1.17.

3.2.5. A morphism of sheaves with generalized transfers commutes with the transfers $\mathrm{Tr}_{F/E}$ and the GW-action hence induces a natural transformation on the Rost-Schmid complex which commutes with the respective maps 3.1.12, 3.1.11 and 3.1.13 defined on the Rost-Schmid complex. After a careful examination of the proof of Theorem 3.2.3, we can thus define a functor

$$\begin{aligned} \tilde{\Gamma}_* : \mathbf{HI}^{\mathrm{gtr}}(k) &\rightarrow \mathbf{HI}^{\mathrm{MW}}(k) \\ M &\mapsto \tilde{\Gamma}_*(M) \end{aligned}$$

which is conservative.

We end this section with a lemma that will be useful later.

Lemma 3.2.6. *Let $M \in \mathbf{HI}^{\mathrm{gtr}}(k)$ be a homotopy sheaf with generalized transfers and let $\psi : E \rightarrow F$ be a finite extension of fields. For any finite model⁵ $f : Y \rightarrow X$ of ψ , we have defined in 3.1.12 a pushforward map*

$$f_* : A^0(Y, M\{\omega_f\}) \rightarrow A^0(X, M).$$

The limit of all such maps over finite models $f : Y \rightarrow X$ defines a map

$$M(F, \omega_{F/k}) \rightarrow M(E, \omega_{E/k})$$

which is equal to the generalized transfer map $\mathrm{Tr}_{F/E}$.

PROOF. This follows from the definitions.

⁵More precisely, X (resp. Y) is an irreducible smooth scheme with function field E (resp. F) and the map $f : Y \rightarrow X$ is finite.

4. Morel's conjecture on Bass-Tate transfers

4.1. Bass-Tate transfers

4.1.1. Let M be a homotopy sheaf and M_{-1} its contraction. We recall the construction of the Bass-Tate transfer maps

$$\mathrm{tr}_\psi = \mathrm{tr}_{F/E} : M_{-1}(F, \omega_{F/k}) \rightarrow M_{-1}(E, \omega_{E/k})$$

defined for any finite map $\psi : E \rightarrow F$ of fields.

Theorem 4.1.2. *Let $M \in \mathbf{HI}(k)$ be a homotopy sheaf. Let F be a field and $F(t)$ the field of rational fractions with coefficients in F in one variable t . We have a split short exact sequence*

$$0 \rightarrow M(F) \xrightarrow{\mathrm{res}} M(F(t)) \xrightarrow{d} \bigoplus_{x \in (\mathbb{A}_F^1)^{(1)}} M_{-1}(\kappa(x), \omega_x) \rightarrow 0$$

where $d = \bigoplus_{x \in (\mathbb{A}_F^1)^{(1)}} \partial_x$ is the usual differential (see 3.1.8).

PROOF. See (Morel, 2012, Theorem 5.38)⁶.

Definition 4.1.3 (Coresidue maps). Keeping the previous notations, the fact that the homotopy sequence is canonically split allows us to define *coresidue maps*

$$\rho_x : M_{-1}(\kappa(x), \omega_x) \rightarrow M(F(t))$$

for any closed points $x \in (\mathbb{A}_F^1)^{(1)}$, satisfying $\partial_x \circ \rho_x = \mathrm{Id}_{\kappa(x)}$ and $\partial_y \circ \rho_x = 0$ for $x \neq y$ where $y \in (\mathbb{A}_F^1)^{(1)}$.

Definition 4.1.4 (Bass-Tate transfers). Let $M \in \mathbf{HI}(k)$ be a homotopy sheaf. Let F be a field and $F(t)$ the field of rational fractions with coefficients in F in one variable t . For $x \in (\mathbb{A}_F^1)^{(1)}$, we define the Bass-Tate transfer

$$\mathrm{tr}_{x/F} : M_{-1}(F(x), \omega_{F(x)/k}) \rightarrow M_{-1}(F, \omega_{F/k})$$

by the formula $\mathrm{tr}_{x/F} = -\partial_\infty \circ \rho_x$.

⁶In fact, Morel does not use twisted sheaves but chooses a canonical generator for each ω_x instead, which is equivalent.

Remark 4.1.5. This notion was defined and studied by Morel in (Morel, 2012, Chapter 4). We use the name *Bass-Tate transfers* because the idea of this definition can be found in Bass and Tate (1973) (where M is the classical Milnor K-theory \mathbf{K}^{MW}) but other names can be found in the literature (e.g. *motivic/geometric transfers*).

Lemma 4.1.6. *Let $M \in \mathbf{HI}(k)$ be a homotopy sheaf. Let $\varphi : E \rightarrow F$ be a field extension, and w be a valuation on F which restricts to a non trivial valuation v on E with ramification e . We have a commutative square*

$$\begin{array}{ccc} M(E) & \xrightarrow{\partial_v} & M_{-1}(\kappa(v), \omega_v) \\ \varphi_* \downarrow & & \downarrow e_\varepsilon \cdot \bar{\varphi}_* \\ M(F) & \xrightarrow{\partial_w} & M_{-1}(\kappa(w), \omega_w) \end{array}$$

where $\bar{\varphi} : \kappa(v) \rightarrow \kappa(w)$ is the induced map and $e_\varepsilon = \sum_{i=1}^e \langle -1 \rangle^{i-1}$.

PROOF. See (Feld, 2021, §3).

We now prove a base change formula (see also (Feld, 2020a, Claim 10) for a similar result). The proof is similar to the original case where M corresponds to Milnor K-theory (see (Bass and Tate, 1973, §1), or (Gille and Szamuely, 2017, §7)).

Lemma 4.1.7. *Let $M \in \mathbf{HI}(k)$ be a homotopy sheaf. Let F/E be a field extension and $x \in (\mathbb{A}_E^1)^{(1)}$ a closed point. Then the following diagram*

$$\begin{array}{ccc} M_{-1}(E(x), \omega_{E(x)/k}) & \xrightarrow{\text{tr}_{x/E}} & M_{-1}(E, \omega_{E/k}) \\ \oplus_y \text{res}_{F(y)/E(x)} \downarrow & & \downarrow \text{res}_{F/E} \\ \bigoplus_{y \mapsto x} M_{-1}(F(y), \omega_{F(y)/k}) & \xrightarrow{\sum_y e_{y,\varepsilon} \text{tr}_{y/F}} & M_{-1}(F, \omega_{F/k}) \end{array}$$

is commutative, where the notation $y \mapsto x$ stands for the closed points of \mathbb{A}_F^1 lying above x , and $e_{y,\varepsilon} = \sum_{i=1}^{e_y} \langle -1 \rangle^{i-1}$ is the quadratic form associated to the ramification index of the valuation v_y extending v_x to $F(t)$.

PROOF. According to Lemma 4.1.6, the following diagram

$$\begin{array}{ccc}
M(E(t)) & \xrightarrow{\partial_x} & M_{-1}(E(x), \omega_x) \\
\text{res}_{F(t)/E(t)} \downarrow & & \downarrow \oplus_y e_{y,\varepsilon} \text{res}_{F(y)/E(x)} \\
M(F(t)) & \xrightarrow{\oplus_y \partial_y} \oplus_{y \mapsto x} & M_{-1}(F(y), \omega_y)
\end{array}$$

is commutative hence for all closed points in \mathbb{P}_F^1 , we have

$$\partial_y(\text{res}_{F(t)/E(t)} \circ \rho_x - (\oplus_y \rho_y) \circ (\oplus_y e_{y,\varepsilon} \text{res}_{F(y)/E(x)})) = 0$$

and so the diagram

$$\begin{array}{ccc}
M(E(t)) & \xleftarrow{\rho_x} & M_{-1}(E(x), \omega_x) \\
\text{res}_{F(t)/E(t)} \downarrow & & \downarrow \oplus_y e_{y,\varepsilon} \text{res}_{F(y)/E(x)} \\
M(F(t)) & \xleftarrow{\oplus_y \rho_y} \oplus_{y \mapsto x} & M_{-1}(F(y), \omega_y)
\end{array}$$

is commutative. Then, we conclude according to the definition of the Bass-Tate transfer maps 4.1.4.

Remark 4.1.8. The multiplicities e_y appearing in the previous lemma are equal to

$$[E(x) : E]_i / [F(y) : F]_i$$

where $[E(x) : E]_i$ is the inseparable degree.

Lemma 4.1.9. *Let $M \in \mathbf{HI}(k)$ be a homotopy sheaf. Let $\varphi : E \rightarrow F = E(x)$ be a simple extension. Then*

1. *For $\langle a \rangle \in \text{GW}(E)$ and $\mu \in M(F, \omega_{F/k})$, one has $\text{tr}_{x/E}(\langle \psi(a) \rangle \cdot \mu) = \langle a \rangle \cdot \text{tr}_{x/E}(\mu)$.*
2. *For $\langle a \rangle \in \text{GW}(F, \omega_{F/k})$ and $\mu \in M(E)$, one has $\text{tr}_{x/E}(\langle a \rangle \cdot \text{res}_{F/E}(\mu)) = \text{tr}_{x/E}(\langle a \rangle) \cdot \mu$.*

PROOF. This follows by GW-linearity from the definitions (see (Bachmann and Yakerson, 2020, Lemma 5.24)).

Definition 4.1.10. Let $M \in \mathbf{HI}(k)$ be a homotopy sheaf. We denote by $M_{\mathbb{Q}}$ the homotopy sheaf defined by

$$X \mapsto M(X) \otimes_{\mathbb{Z}} \mathbb{Q}$$

and by $M_{-1, \mathbb{Q}}$ the homotopy sheaf $(M_{\mathbb{Q}})_{-1}$.

Corollary 4.1.11. *Let $M \in \mathbf{HI}(k)$ be a homotopy sheaf. Let $\varphi : E \rightarrow F = E(x)$ be a simple extension. The kernel of the restriction map $\text{res}_{F/E} : M_{-1}(E) \rightarrow M_{-1}(F)$ is killed by $\text{tr}_{x/E}(1)$. In particular: if the Hopf map $\boldsymbol{\eta}$ acts trivially on M_{-1} , then the restriction map $\text{res}_{F/E} : M_{-1, \mathbb{Q}}(E) \rightarrow M_{-1, \mathbb{Q}}(F)$ is injective.*

PROOF. Let $x \in \ker(\text{res}_{F/E})$. The previous projection formula shows that $\text{tr}_{x/E}(1) \cdot x = 0$, thus the first assertion. A priori, $\text{tr}_{x/E}(1)$ is in $\text{GW}(E)$, i.e. an element of the form $\sum_{i=1}^n \langle a_i \rangle$ (where $n = [F : E]$, $a_i \in E^\times$ and $\langle a_i \rangle = 1 + \boldsymbol{\eta}[a_i]$); but if we assume moreover that $\boldsymbol{\eta}$ acts trivially, then the action of $\text{tr}_{x/E}(1)$ on $M_{-1, \mathbb{Q}}(E)$ is the multiplication by n (which is a nonzero natural number) and thus $x = 0$, which proves the second assertion.

Definition 4.1.12. Let $F = E(x_1, x_2, \dots, x_r)$ be a finite extension of a field E and consider the chain of subfields

$$E \subset E(x_1) \subset E(x_1, x_2) \subset \dots \subset E(x_1, \dots, x_r) = F.$$

Define by induction:

$$\text{tr}_{x_1, \dots, x_r/E} = \text{tr}_{x_r/E(x_1, \dots, x_{r-1})} \circ \dots \circ \text{tr}_{x_2/E(x_1)} \circ \text{tr}_{x_1/E}$$

Conjecture 4.1.13 (Morel conjecture). Keeping the previous notations, let $F = E(x_1, x_2, \dots, x_r)$ be a finite extension of a field E . The map

$$\text{tr}_{x_1, \dots, x_r/E} : M_{-1}(F, \omega_{F/k}) \rightarrow M_{-1}(E, \omega_{E/k})$$

does not depend on the choice of the generating system (x_1, \dots, x_r) .

Remark 4.1.14. This was claimed by Morel in (Morel, 2012, Remark 4.31) and (Morel, 2011, Remark 5.10) (see also (Bachmann, 2020, Remark 4.3) for a similar conjecture). Historically, a similar conjecture was made by Bass and Tate in Bass and Tate (1973) for the case where $M = \mathbf{K}^{\text{MW}}$ is the Milnor K-theory (this claim was proved only a decade later by Kato Kato (1980)).

Proposition 4.1.15 (Projection formulas). *Let $\varphi : E \rightarrow F = E(x_1, x_2, \dots, x_r)$ be a finite extension. Then*

1. For $\langle a \rangle \in \text{GW}(E)$ and $\mu \in M(F, \omega_{F/k})$, one has $\text{tr}_{x_1, \dots, x_r/E}(\langle \psi(a) \rangle \cdot \mu) = \langle a \rangle \cdot \text{tr}_{x_1, \dots, x_r/E}(\mu)$.
2. For $\langle a \rangle \in \text{GW}(F, \omega_{F/k})$ and $\mu \in M(E)$, one has $\text{tr}_{x_1, \dots, x_r/E}(\langle a \rangle \cdot \text{res}_{F/E}(\mu)) = \text{tr}_{x_1, \dots, x_r/E}(\langle a \rangle) \cdot \mu$.

PROOF. This is immediate by induction on r according to Lemma 4.1.9.

Theorem 4.1.16 (Strong R1c). *Let $M \in \mathbf{HI}(k)$ be a homotopy sheaf. Let E be a field, F/E a finite field extension and L/E an arbitrary field extension. Write $F = E(x_1, \dots, x_r)$ with $x_i \in F$, $R = F \otimes_E L$ and $\psi_{\mathfrak{p}} : R \rightarrow R/\mathfrak{p}$ the natural projection defined for any $\mathfrak{p} \in \text{Spec}(R)$. Then the diagram*

$$\begin{array}{ccc}
 M_{-1}(F, \omega_{F/k}) & \xrightarrow{\text{tr}_{x_1, \dots, x_r/E}} & M_{-1}(E, \omega_{E/k}) \\
 \oplus_{\mathfrak{p}} \text{res}_{(R/\mathfrak{p})/F} \downarrow & & \downarrow \text{res}_{L/E} \\
 \oplus_{\mathfrak{p} \in \text{Spec}(R)} M_{-1}(R/\mathfrak{p}, \omega_{(R/\mathfrak{p})/k}) & \xrightarrow{\sum_{\mathfrak{p}} (m_{\mathfrak{p}})_{\varepsilon} \text{tr}_{\psi_{\mathfrak{p}}(a_1), \dots, \psi_{\mathfrak{p}}(a_r)/L}} & M_{-1}(L, \omega_{L/k})
 \end{array}$$

is commutative where $(m_{\mathfrak{p}})_{\varepsilon}$ is the quadratic form associated to the length of the localized ring $R_{(\mathfrak{p})}$ (see Notation 1.2).

PROOF. We prove the theorem by induction. For $r = 1$, this is Lemma 4.1.7. Write $E(x_1) \otimes_E L = \prod_j R_j$ for some Artin local L -algebras R_j , and decompose the finite dimensional L -algebra $F \otimes_{E(x_1)} R_j$ as $F \otimes_{E(x_1)} R_j = \prod_i R_{ij}$ for some local L -algebras R_{ij} . We have $F \otimes_E L \simeq \prod_{i,j} R_{ij}$. Denote by L_j (resp. L_{ij}) the residue fields of the Artin local L -algebras R_j (resp. R_{ij}), and m_j (resp. m_{ij}) for their geometric multiplicity. We can conclude as the following diagram commutes

$$\begin{array}{ccccc}
 M_{-1}(F, \omega_{F/k}) & \xrightarrow{\text{tr}_{x_1, \dots, x_r/E}} & M_{-1}(E(x_1), \omega_{E(x_1)/k}) & \xrightarrow{\text{tr}_{x_1/E}} & M_{-1}(E, \omega_{E/k}) \\
 \oplus_{ij} \text{res}_{L_{ij}/F} \downarrow & & \downarrow \oplus \text{res}_{L_j/E(x_1)} & & \downarrow \text{res}_{L/E} \\
 \oplus_{ij} M_{-1}(L_{ij}, \omega_{L_{ij}/k}) & \xrightarrow{\sum_{ij} (m_{ij} m_j^{-1})_{\varepsilon} \text{tr}_{\psi_{ij}(x_1), \dots, \psi_{ij}(x_r)/L_j}} & \oplus_j M_{-1}(L_j, \omega_{L_j/k}) & \xrightarrow{\sum_j (m_j)_{\varepsilon} \text{tr}_{\psi_j(x_1)/L}} & M_{-1}(L, \omega_{L/k})
 \end{array}$$

since both squares are commutative by the inductive hypothesis and the multiplicity formula $(mn)_{\varepsilon} = m_{\varepsilon} n_{\varepsilon}$ for any natural numbers m, n .

Proposition 4.1.17. *Let $M \in \mathbf{HI}(k)$ be a homotopy sheaf. Let $E \rightarrow F$ be a field extension. If the Hopf map η acts trivially on M_{-1} , then the restriction map $\text{res}_{F/E} : M_{-1, \mathbb{Q}}(E) \rightarrow M_{-1, \mathbb{Q}}(F)$ is injective.*

PROOF. We may assume that F/E is finitely generated. By induction on the number of generators of F over E , we may assume that $F = E(x)$ is generated by a single element $x \in F$. If x is algebraic over E , we know from Corollary 4.1.11 that $\text{res}_{F/E}$ is killed by $\text{tr}_{x/E}(1)$. Since the Hopf map acts trivially, the action of $\text{tr}_{x/E}(1)$ is given by the multiplication by $[F : E]$ and we obtain the result. If x is transcendental over E , then we know (thanks to Theorem 4.1.2) that $M_{-1}(E)$ is a direct summand in $M_{-1}(F)$.

Corollary 4.1.18. *Let $M \in \mathbf{HI}(k)$ be a homotopy sheaf. Let F/E be a finite field extension, and let x_1, \dots, x_r and y_1, \dots, y_s be two generating system for F/E . If the Hopf map η acts trivially on M_{-1} , then*

$$\text{tr}_{x_1, \dots, x_r/E} = \text{tr}_{y_1, \dots, y_s/E}$$

seen as morphism from $M_{-1, \mathbb{Q}}(F)$ to $M_{-1, \mathbb{Q}}(E)$.

PROOF. We apply Theorem 4.1.16 two times with $F = E(x_1, \dots, x_r)$ and $F = E(y_1, \dots, y_s)$, and with $L = \overline{E}$ an algebraic closure of E . Hence $\text{res}_{L/E} \circ \text{tr}_{x_1, \dots, x_r/E} = \text{res}_{L/E} \circ \text{tr}_{y_1, \dots, y_s/E}$ and we end with Proposition 4.1.17.

4.1.19. In the following, we assume that 2 is invertible. The proof of the next theorem uses a technique which consists in splitting an object into a (+)-part and a (-)-part. Usually, the splitting is only discussed for \mathbb{P}^1 -spectra (see e.g. (Cisinski and Déglise, 2019, §16.2.1)) but similar results hold in our more general context.

The isomorphism

$$\begin{array}{c} \mathbb{G}_m \rightarrow \mathbb{G}_m \\ t \rightarrow t^{-1} \end{array}$$

defines (following (Morel, 2003, Lemma 6.1.1)) an element ε in $\text{End}_{\mathbf{SH}^{S^1}(k, \mathbb{Q})}(\mathbb{G}_m)$ which satisfies $\varepsilon^2 = 1$. We define two projectors of \mathbb{G}_m in the category $\mathbf{SH}^{S^1}(k, \mathbb{Q})$:

$$e_+ = \frac{1-\varepsilon}{2} \text{ and } e_- = \frac{1+\varepsilon}{2}.$$

As the triangulated category $\mathbf{SH}^{S^1}(k, \mathbb{Q})$ is pseudo-abelian, we can define two objects as follows:

$$\mathbb{G}_{m+} = \text{Im}(e_+) \text{ and } \mathbb{G}_{m-} = \text{Im}(e_-).$$

Then for any object $M \in \mathbf{HI}(k)$, we set

$$(M_{-1, \mathbb{Q}})_+ = \underline{\mathcal{H}om}(\mathbb{G}_{m+}, M_{\mathbb{Q}}) \text{ and } (M_{-1, \mathbb{Q}})_- = \underline{\mathcal{H}om}(\mathbb{G}_{m-}, M_{\mathbb{Q}})$$

We have a decomposition

$$M_{-1, \mathbb{Q}} = (M_{-1, \mathbb{Q}})_+ \oplus (M_{-1, \mathbb{Q}})_-.$$

Finally, we also recall that $\eta\varepsilon = \eta$ (according to (Morel, 2003, §6.2.3)).

Theorem 4.1.20. *Assume that 2 is invertible. Let $M \in \mathbf{HI}(k)$ be a homotopy sheaf. The sheaf $M_{-1, \mathbb{Q}}$ has functorial Bass-Tate transfer maps (i.e. Conjecture 4.1.13 holds for $M_{\mathbb{Q}}$).*

PROOF. The sheaf $M_{-1, \mathbb{Q}}$ splits into two sheaves $(M_{-1, \mathbb{Q}})_+$ and $(M_{-1, \mathbb{Q}})_-$. On one hand, the Hopf map η acts trivially on $M_{-1, \mathbb{Q}}^+$ hence there is a structure of functorial transfers thanks to Corollary 4.1.18. On the other hand, we have (according to (Morel, 2012, Lemma 3.43)):

$$e_- = \frac{1+\varepsilon}{2} = \frac{1-\langle -1 \rangle}{2} = \frac{1-(1+\eta[-1])}{2} = -\frac{\eta[-1]}{2}$$

which is equal to $\text{Id}_{\mathbb{G}_{m-}}$ on the minus part. Since \mathbb{G}_m -stabilization (from S^1 -spectra to S^1 -spectra) induces an isomorphism on the endomorphism groups of positive powers of \mathbb{G}_m (one always gets GW, see also (Morel, 2012, Corollary 6.43)), one can check \mathbb{P}^1 -stably that η induces an isomorphism $\mathbb{G}_{m-} \wedge \mathbb{G}_m \rightarrow \mathbb{G}_{m-}$ and thus

$$(M_{-1, \mathbb{Q}})_- \simeq \underline{\mathcal{H}om}(\mathbb{G}_{m-}, M_{\mathbb{Q}}) \simeq \underline{\mathcal{H}om}(\mathbb{G}_{m-} \wedge \mathbb{G}_m, M_{\mathbb{Q}}) \simeq (M_{-2, \mathbb{Q}})_-.$$

The latter has a structure of functorial transfers according to (Morel, 2012, Theorem 4.27). Hence the result.

We summarize the previous results in the following theorem.

Theorem 4.1.21. *Let $M \in \mathbf{HI}(k)$ be a homotopy sheaf. Then:*

1. *Assume that 2 is invertible. The rational contracted homotopy sheaf $M_{-1, \mathbb{Q}}$ is a homotopy sheaf with generalized transfers.*

2. Assuming Conjecture 4.1.13, the contracted homotopy sheaf M_{-1} is a homotopy sheaf with generalized transfers.

PROOF. The GW-action is defined in 2.2.3, functoriality of transfers eR1b follows from Corollary 4.1.18 or Conjecture 4.1.13, the base change eR1c is Theorem 4.1.16, the projection formulas eR2 are proved in Proposition 4.1.15 and the compatibility axiom eR3b can be deduced from (Morel, 2012, Theorem 5.19).

4.2. Unicity of transfers

The goal of this subsection is to prove Theorem 4.2.3 which asserts that the structure of Bass-Tate transfer maps on a contracted homotopy sheaf M_{-1} is, in some sense, unique.

Lemma 4.2.1 (eR3c and eR3d). *Let $M \in \mathbf{HI}(k)$ be a homotopy sheaf. Let $\iota : E \rightarrow F$ be an extension of fields and w a valuation on F that restricts to the trivial valuation on E . Then the composition*

$$M(E) \xrightarrow{\iota^*} M(F) \xrightarrow{\partial_w} M_{-1}(\kappa(w), \omega_w)$$

is zero. Moreover, let $\bar{\iota} : E \rightarrow \kappa(w)$ be the induced map. For any prime π of w , the following diagram

$$\begin{array}{ccc} M_{-1}(E) & \xrightarrow{\iota^*} & M_{-1}(F) \\ \bar{\iota}^* \downarrow & & \downarrow [\pi] \\ M_{-1}(\kappa(w)) & \xleftarrow{\Theta \circ \partial_w} & M(F) \end{array}$$

is commutative (where Θ is the canonical isomorphism induced by the trivialization of ω_w through the choice of π).

PROOF. The first identity is deduced from the long exact sequence 2.2.5. The commutative square follows from (Bachmann and Yakerson, 2020, Corollary 5.23).

Lemma 4.2.2 (eR3e). *Let E be a field over k with a valuation v and let u be a unit of v . Then*

$$\partial_v \circ [u] = \varepsilon[u] \circ \partial_v$$

where ε is $-\langle -1 \rangle$ as usual.

PROOF. See (Morel, 2012, Lem 5.10).

Theorem 4.2.3. *Let $M \in \mathbf{HI}^{\text{gtr}}(k)$ be a homotopy sheaf with generalized transfers. We have for any simple extension $\psi : F \rightarrow F(x)$:*

- (a) *A generalized transfer map $\text{Tr}_\psi : M_{-1}(F(x), \omega_{F(x)/k}) \rightarrow M_{-1}(F, \omega_{F/k})$ induced by the structure of a homotopy sheaf with generalized transfers on M (see also Remark 3.1.5).*
- (b) *A Bass-Tate transfer map $\text{tr}_{x/F} : M_{-1}(F(x), \omega_{F(x)/k}) \rightarrow M_{-1}(F, \omega_{F/k})$ defined in 4.1.4.*

Then the two transfer maps coincide : $\text{Tr}_\psi = \text{tr}_{x/F}$.

PROOF. Fix a field F and $F(t)$ the field of rational fractions with coefficients in F in one variable t , and fix a simple extension $\psi : F \rightarrow F(x)$. Consider the canonical inclusion $\iota : F(x) \rightarrow F(x)(t)$ and define

$$\Phi^x : M_{-1}(F(x), \omega_{F(x)/k}) \rightarrow M_{-1}(F(x)(t), \omega_{F(x)(t)/k})$$

as the composite $\Phi^x = \text{Tr}_{F(x)(t)/F(x)} \circ [t - \iota(x)] \circ \iota^*$ where Tr denotes the generalized transfers of point **(a)**. A combination of eR3b, Lemma 4.2.1 and Lemma 4.2.2 shows that

$$\begin{aligned} \partial_x \circ \Phi^x &= \text{Id}, \\ -\partial_\infty \circ \Phi^x &= \text{Tr}_{F(x)/F}, \end{aligned}$$

which is exactly the definition of the Bass-Tate transfers $\text{tr}_{F(x)/F}$.

5. MW-homotopy sheaves

5.1. Sheaves with MW-transfers

In this subsection, we recall the basic definition of sheaves with MW-transfers in order to fix the notations. We follow the presentation of (Bachmann et al., 2020, Chapter 2).

5.1.1. Let X and Y be smooth schemes over k and let $T \subset X \times Y$ be a closed subset. Any irreducible component of T maps to an irreducible component of X through the projection $X \times Y \rightarrow X$.

Definition 5.1.2. If, when T is endowed with its reduced structure, the projection map $T \rightarrow X$ is finite and surjective for every irreducible component of T , we say that T is an admissible subset of $X \times Y$. We denote by $\mathcal{A}(X, Y)$ the set of admissible subsets of $X \times Y$, partially ordered by inclusions.

5.1.3. If Y is equidimensional, $d = \dim Y$ and $p_Y : X \times Y \rightarrow Y$ is the projection, we define a covariant functor

$$\mathcal{A}(X, Y) \rightarrow \mathcal{A}b$$

by associating to each admissible subset $T \in \mathcal{A}(X, Y)$ the group

$$\widetilde{\mathrm{CH}}_T^d(X \times Y, p_Y^* \omega_{Y/k}) = H_T^d(X \times Y, \mathbf{K}_d^{\mathrm{MW}}\{p_Y^* \omega_{Y/k}\})$$

(see (Fasel, 2020, Definition 2.5)) and to each morphism $T' \subset T$ the extension of support morphism

$$\widetilde{\mathrm{CH}}_{T'}^d(X \times Y, p_Y^* \omega_{Y/k}) \rightarrow \widetilde{\mathrm{CH}}_T^d(X \times Y, p_Y^* \omega_{Y/k})$$

and, using that functor, we set

$$\widetilde{\mathrm{Cor}}_k(X, Y) = \mathrm{colim}_{T \in \mathcal{A}(X, Y)} \widetilde{\mathrm{CH}}_T^d(X \times Y, p_Y^* \omega_{Y/k}).$$

If Y is not equidimensional, then $Y = \bigsqcup_j Y_j$ with each Y_j equidimensional and we set

$$\widetilde{\mathrm{Cor}}_k(X, Y) = \prod_j \widetilde{\mathrm{Cor}}_k(X, Y_j).$$

By additivity of Chow-Witt groups, if $X = \bigsqcup_i X_i$ and $Y = \bigsqcup_j Y_j$ are the respective decompositions of X and Y in irreducible components, we have

$$\widetilde{\mathrm{Cor}}_k(X, Y) = \prod_{i,j} \widetilde{\mathrm{Cor}}_k(X_i, Y_j).$$

Remark 5.1.4. In the sequel, we will simply write ω_Y in place of $p_Y^* \omega_{Y/k}$.

Example 5.1.5. Let X be a smooth scheme of dimension d . Then

$$\widetilde{\mathrm{Cor}}_k(\mathrm{Spec}(k), X) = \bigoplus_{x \in X^{(d)}} \widetilde{\mathrm{CH}}_{\{x\}}^d(X, \omega_X) = \bigoplus_{x \in X^{(d)}} \mathrm{GW}(\kappa(x), \omega_{\kappa(x)/k}).$$

On the other hand, $\widetilde{\mathrm{Cor}}_k(X, \mathrm{Spec}(k)) = \widetilde{\mathrm{CH}}^0(X) = \mathbf{K}_0^{\mathrm{MW}}(X)$ (recall 2.1.19) for any smooth scheme X .

5.1.6. The group $\widetilde{\text{Cor}}_k(X, Y)$ admits an alternate description which is often useful. Let X and Y be smooth schemes, with Y equidimensional. For any closed subscheme $T \subset X \times Y$ of codimension $d = \dim Y$, we have an inclusion

$$\widetilde{\text{CH}}_T^d(X \times Y, \omega_{Y/k}) \subset \bigoplus_{x \in (X \times Y)^{(d)}} \mathbf{K}_0^{\text{MW}}(\kappa(x), \det(\omega_x \otimes (\omega_{Y/k})_x))$$

and thus

$$\begin{aligned} \widetilde{\text{Cor}}_k(X, Y) &= \bigcup_{T \in \mathcal{A}(X, Y)} \widetilde{\text{CH}}_T^d(X \times Y, \omega_{Y/k}) \subset \\ &\quad \bigoplus_{x \in (X \times Y)^{(d)}} \mathbf{K}_0^{\text{MW}}(\kappa(x), \det(\omega_x \otimes (\omega_{Y/k})_x)). \end{aligned}$$

In general, the inclusion

$$\widetilde{\text{Cor}}_k(X, Y) \subset \bigoplus_{x \in (X \times Y)^{(d)}} \mathbf{K}_0^{\text{MW}}(\kappa(x), \det(\omega_x \otimes (\omega_{Y/k})_x))$$

is strict as shown by Example 5.1.5. As an immediate consequence of this description, we see that the map

$$\widetilde{\text{CH}}_T^d(X \times Y, \omega_Y) \rightarrow \widetilde{\text{Cor}}_k(X, Y)$$

is injective for any $T \in \mathcal{A}(X, Y)$.

5.1.7 (Composition of finite MW-correspondences). Let X, Y and Z be smooth schemes of respective dimension d_X, d_Y and d_Z , with X and Y connected. Let $V \in \mathcal{A}(X, Y)$ and $\mathcal{A}(Y, Z)$ be admissible subsets. If $\beta \in \widetilde{\text{CH}}_V^{d_Y}(X \times Y, \omega_{Y/k})$ and $\alpha \in \widetilde{\text{CH}}_T^{d_Z}(Y \times Z, \omega_{Z/k})$ are two cycles, then the expression

$$\alpha \circ \beta = (q_{XY})_*[(q_{YZ})^* \beta \cdot (p_{XY})^* \alpha]$$

is well-defined and yields a composition

$$\circ : \widetilde{\text{Cor}}_k(X, Y) \times \widetilde{\text{Cor}}_k(Y, Z) \rightarrow \widetilde{\text{Cor}}_k(X, Z)$$

which is associative (see (Bachmann et al., 2020, Ch.2, §4.2)).

Definition 5.1.8. The category of finite MW-correspondences over k is by definition the category $\widetilde{\text{Cor}}_k$ whose objects are smooth schemes and whose morphisms are the elements of abelian groups $\widetilde{\text{Cor}}_k(X, Y)$.

Remark 5.1.9. The category $\widetilde{\text{Cor}}_k$ is a symmetric monoidal additive category (see (Bachmann et al., 2020, Chapter 2, Lemma 4.4.2)).

Definition 5.1.10. A presheaf with MW-transfers is a contravariant additive functor $\widetilde{\text{Cor}}_k \rightarrow \mathcal{A}b$. A (Nisnevich) sheaf with MW-transfers is a presheaf with MW-transfers such that its restriction to Sm_k via the graph functor is a Nisnevich sheaf. We denote by $\widetilde{\text{PSh}}(k)$ (resp. $\widetilde{\text{Sh}}(k)$) the category of presheaves (resp. sheaves) with MW-transfers and by $\mathbf{HI}^{\text{MW}}(k)$ the category of homotopy sheaves with MW-transfers (also called *MW-homotopy sheaves*).

Example 5.1.11. For any $j \in \mathbb{Z}$, the contravariant functor $X \mapsto \mathbf{K}_j^{\text{MW}}(X)$ is a presheaf on $\widetilde{\text{Cor}}_k$.

5.1.12. PUSHFORWARDS. Let X and Y be two smooth schemes of dimension d and let $f : X \rightarrow Y$ be a finite morphism such that any irreducible component of X surjects to the irreducible component of Y it maps to. Assume that we have an orientation (\mathcal{L}, ψ) of ω_f , that is an isomorphism $\psi : \mathcal{L} \otimes \mathcal{L} \rightarrow \omega_f$ of line bundles. We define a finite correspondence $\alpha(f, \mathcal{L}, \psi) \in \widetilde{\text{Cor}}_k(Y, X)$. Let $\gamma'_f : X \rightarrow Y \times X$ be the transpose of the graph of f . Since X is an admissible subset of $Y \times X$, we have a transfer map

$$(\gamma'_f)_* : \mathbf{K}_0^{\text{MW}}(X, \omega_f) \rightarrow \widetilde{\text{CH}}_X^d(Y \times X, \omega_{X/k}) \rightarrow \widetilde{\text{Cor}}_k(Y, X).$$

The map ψ yields an isomorphism $\mathbf{K}_0^{\text{MW}}(X) \rightarrow \mathbf{K}_0^{\text{MW}}(X, \omega_f)$. We define the finite MW-correspondence $\alpha(f, \mathcal{L}, \psi)$ as the image of $\langle 1 \rangle$ under the composite

$$\mathbf{K}_0^{\text{MW}}(X) \rightarrow \mathbf{K}_0^{\text{MW}}(X, \omega_f) \rightarrow \widetilde{\text{CH}}_X^d(Y \times X, \omega_{X/k}) \rightarrow \widetilde{\text{Cor}}_k(Y, X).$$

Now let $M \in \mathbf{HI}^{\text{MW}}(k)$ be a homotopy sheaf with MW-transfers. Denote by $(M \otimes \omega_f)_X$ (resp. M_Y) the canonical (twisted) sheaf associated to M defined on the Zariski site X_{Zar} (resp. Y_{Zar}) introduced in 2.2.4 and define a natural transformation

$$f_* : f_*(M \otimes \omega_f)_X \rightarrow M_Y$$

by taking (as in 2.2.4) the sheafification of the natural transformation of presheaves

$$\begin{aligned} V \in Y_{\text{Zar}} &\mapsto (M(f^{-1}(V), \omega_{f|_{f^{-1}(V)}}) \rightarrow M(V)) \\ (\mu \otimes l) &\mapsto \alpha(f, \psi_l, L_l)^*(\mu) \end{aligned}$$

where (ψ_l, \mathcal{L}_l) is the orientation of $\omega_{f|_{f^{-1}(V)}}$ associated to $l \in \omega_{f|_{f^{-1}(V)}}^\times$. Taking global sections, this leads in particular to a map

$$M({}^t f) : M(X, \omega_f) \rightarrow M(Y)$$

for any finite morphism $f : X \rightarrow Y$.

We can check the following propositions.

Proposition 5.1.13. *Let $M \in \mathbf{HI}^{\text{MW}}(k)$ and consider two finite morphisms $X \xrightarrow{f} Y \xrightarrow{g} Z$ of smooth schemes. Then*

$$M({}^t(g \circ f)) = M({}^t g) \circ M({}^t f).$$

PROOF. Keeping the previous notations, if (\mathcal{L}', ψ') is an orientation of ω_g , then $(\mathcal{L} \otimes f^* \mathcal{L}', \psi \otimes f^* \psi')$ is an orientation of $\omega_{g \circ f} = \omega_f \otimes f^* \omega_g$, and we have $\alpha(f, \mathcal{L}, \psi) \circ \alpha(g, \mathcal{L}', \psi') = \alpha(g \circ f, \mathcal{L} \otimes f^* \mathcal{L}', \psi \otimes f^* \psi')$.

Proposition 5.1.14. *Let $M \in \mathbf{HI}^{\text{MW}}(k)$ be a homotopy sheaf with MW-transfers. Let $i : Z \rightarrow X$ and $i' : T \rightarrow Y$ be two closed immersions and let $f : Y \rightarrow X$ be a finite morphism. The following diagram*

$$\begin{array}{ccc} M(Y - T) & \xrightarrow{\partial_{i'}} & M_{-1}(T, \omega_{T/Y}) \\ M({}^t f) \downarrow & & \downarrow M({}^t f) \\ M(X - Z) & \xrightarrow{\partial_i} & M_{-1}(Z, \omega_{Z/X}) \end{array}$$

is commutative.

PROOF. Since we can identify $M_{-1}(T, \omega_{i'})$ with $H_T^1(Y, M)$ (and $M_{-1}(T, \omega_i)$ with $H_Z^1(X, M)$), the result follows from the fact that MW-transfers act on cohomology with support and the localization long exact sequence is functorial (see also (Déglise et al., 2021, Proposition 2.2.11) for a similar result).

5.1.15. TENSOR PRODUCTS. Let X_1, X_2, Y_1, Y_2 be smooth schemes over $\text{Spec } k$. Let $d_1 = \dim Y_1$ and $d_2 = \dim Y_2$. Let $\alpha_1 \in \widetilde{\text{CH}}_{T_1}^{d_1}(X_1 \times Y_1, \omega_{Y_1/k})$ and $\alpha_2 \in \widetilde{\text{CH}}_{T_2}^{d_2}(X_2 \times Y_2, \omega_{Y_2/k})$ for some admissible subsets $T_i \subset X_i \times Y_i$. The exterior product defined in (Fasel, 2008, §4) gives a cycle

$$(\alpha_1 \times \alpha_2) \in \widetilde{\text{CH}}_{T_1 \times T_2}^{d_1+d_2}(X_1 \times Y_1 \times X_2 \times Y_2, p_{Y_1}^* \omega_{Y_1/k} \otimes p_{Y_2}^* \omega_{Y_2/k})$$

where $p_{Y_i} : X_1 \times Y_1 \times X_2 \times Y_2 \rightarrow Y_i$ is the canonical projection to the corresponding factor. Let $\sigma : X_1 \times Y_1 \times X_2 \times Y_2 \rightarrow X_1 \times X_2 \times Y_1 \times Y_2$ be the transpose isomorphism. Applying σ_* , we get a cycle

$$\sigma_*(\alpha_1 \times \alpha_2) \in \widetilde{\mathrm{CH}}_{\sigma(T_1 \times T_2)}^{d_1+d_2}(X_1 \times X_2 \times Y_1 \times Y_2, p_{Y_1}^* \omega_{Y_1/k} \otimes p_{Y_2}^* \omega_{Y_2/k}).$$

Since $p_{Y_1}^* \omega_{Y_1/k} \otimes p_{Y_2}^* \omega_{Y_2/k} = \omega_{Y_1 \times Y_2/k}$, it is straightforward to check that $\sigma(T_1 \times T_2)$ is finite and surjective over $X_1 \times X_2$. Thus $\sigma_*(\alpha_1 \times \alpha_2)$ defines a finite MW-correspondence between $X_1 \times X_2$ and $Y_1 \times Y_2$.

Definition 5.1.16. Let X_1, X_2, Y_1, Y_2 be smooth schemes over $\mathrm{Spec} k$, and $\alpha_1 \in \widetilde{\mathrm{Cor}}_k(X_1, Y_1)$ and $\alpha_2 \in \widetilde{\mathrm{Cor}}_k(X_2, Y_2)$ two MW-correspondences. We define their tensor products as $X_1 \otimes X_2 = X_1 \times X_2$ and $\alpha_1 \otimes \alpha_2 = \sigma_*(\alpha_1 \times \alpha_2)$.

5.1.17. We denote by $\tilde{c}(X) : Y \mapsto \widetilde{\mathrm{Cor}}_k(Y, X)$ the representable presheaf associated to a smooth scheme X (be careful that this is not a Nisnevich sheaf in general). The category of MW-presheaves is an abelian Grothendieck category with a unique symmetric monoidal structure such that the Yoneda embedding

$$\widetilde{\mathrm{Cor}}_k \rightarrow \widetilde{\mathrm{Sh}}(k), X \mapsto \tilde{a}\tilde{c}(X)$$

is symmetric monoidal (where \tilde{a} is the sheafification functor, see (Bachmann et al., 2020, Chapter 3, §1.2.7)). The tensor product is denoted by $\otimes_{\mathbf{HI}^{\mathrm{MW}}}$ and commutes with colimits (hence the monoidal structure is closed, see (Bachmann et al., 2020, Chapter 3, §1.2.14)).

5.2. Structure of generalized transfers

In this section, we study the category of MW-homotopy sheaves. For any MW-homotopy sheaf we construct a canonical structure of generalized transfers (see Definition 3.1.1).

5.2.1. Let $M \in \mathbf{HI}^{\mathrm{MW}}(k)$ be a homotopy sheaf with MW-transfers. We denote by $\tilde{\Gamma}^*(M)$ the homotopy sheaf M equipped with its structure of GW-module coming from its structure of MW-transfers, and we define generalized transfers as follows. Let $\psi : E \rightarrow F$ be a finite extension of fields. Consider a smooth model (X, x) (resp. (Y, y)) of E/k (resp. F/k) such that ψ corresponds to a map $Y_y \rightarrow X_x$. We may assume that this map is induced by a finite morphism $f : Y \rightarrow X$. We consider the pushforward on the MW-homotopy sheaf $\tilde{\Gamma}^*(M)$ with respect to the finite morphism f defined in 5.1.12 and take the limit over all model of F/E so that we obtain a morphism

$$\psi^* : \tilde{\Gamma}^*(M)(F, \omega_{F/k}) \rightarrow \tilde{\Gamma}^*(M)(E, \omega_{E/k})$$

of abelian groups. This defines a homotopy sheaf $\tilde{\Gamma}^*(M)$ canonically isomorphic to M (as presheaves) and equipped with a structure of transfers eD2.

Theorem 5.2.2. *Keeping the previous notations, the transfer maps ψ^* defines a structure of generalized transfers (see Definition 3.1.1) on the homotopy sheaf $\tilde{\Gamma}^*(M)$.*

PROOF. The functoriality property eR1b results from Proposition 5.1.13.

According to Proposition 4.1.9 and Theorem 4.1.16, the projection formulas eR2 and the base change rule eR1c are true for any contracted homotopy sheaf hence we only have to prove that $\tilde{\Gamma}^*(M)$ is a contracted homotopy sheaf:

Lemma 5.2.3. *Let $M \in \mathbf{HI}^{\text{MW}}(k)$ a MW-homotopy sheaf. Then there is a canonical isomorphism*

$$\tilde{\Gamma}^*(M) \simeq (\mathbb{G}_m \otimes_{\mathbf{HI}^{\text{MW}}} \tilde{\Gamma}^*(M))_{-1}$$

of homotopy sheaves which is compatible with the generalized transfers structure in the sense that the diagram

$$\begin{array}{ccc} \tilde{\Gamma}^*(M)(E(x)) & \xrightarrow{\psi^*} & \tilde{\Gamma}^*(M)(E) \\ \downarrow \simeq & & \downarrow \simeq \\ (\mathbb{G}_m \otimes_{\mathbf{HI}^{\text{MW}}} \tilde{\Gamma}^*(M))_{-1}(E(x)) & \xrightarrow{\text{tr}_{x/E}} & (\mathbb{G}_m \otimes_{\mathbf{HI}^{\text{MW}}} \tilde{\Gamma}^*(M))_{-1}(E) \end{array}$$

is commutative for any simple extension $\psi : E \rightarrow E(x)$ of fields, where $\text{tr}_{x/E}$ is the Bass-Tate transfer map defined in 4.1.4.

PROOF. The isomorphism $\tilde{\Gamma}^*(M) \simeq (\mathbb{G}_m \otimes_{\mathbf{HI}^{\text{MW}}} \tilde{\Gamma}^*(M))_{-1}$ is an equivalent reformulation of the cancellation theorem (Fasel and Østvær, 2017, Theorem 4.0.1). The second assertion is a corollary of Theorem 4.2.3.

Still denoting by $M \in \mathbf{HI}^{\text{MW}}(k)$ a MW-homotopy sheaf, we need to check that $\tilde{\Gamma}^*(M)$ satisfies eR3b where $M \in \mathbf{HI}^{\text{MW}}(k)$, which can be deduced from the definitions and Proposition 5.1.14. This concludes the proof of Theorem 5.2.2.

5.2.4. As in 3.2.5, we see that $\tilde{\Gamma}^*$ defines a functor

$$\begin{aligned}\tilde{\Gamma}^* : \mathbf{HI}^{\text{MW}}(k) &\rightarrow \mathbf{HI}^{\text{gtr}}(k) \\ M &\mapsto \tilde{\Gamma}^*(M)\end{aligned}$$

which is conservative.

Theorem 5.2.5. *Keeping the notations of 3.2.5 and 5.2.4, the functors*

$$\mathbf{HI}^{\text{MW}}(k) \begin{array}{c} \xrightarrow{\tilde{\Gamma}^*} \\ \xleftarrow{\tilde{\Gamma}_*} \end{array} \mathbf{HI}^{\text{gtr}}(k)$$

form an equivalence of categories.

PROOF. First, let $M \in \mathbf{HI}^{\text{gtr}}(k)$ be a homotopy sheaf with generalized transfers. For any smooth scheme X , we have a canonical isomorphism

$$a_X : \tilde{\Gamma}^* \tilde{\Gamma}_*(M)(X) \rightarrow M(X)$$

which is compatible with pullback maps and the GW-action. Compatibility with the generalized transfers eD2 results from Lemma 3.2.6.

Second, let $M \in \mathbf{HI}^{\text{MW}}(k)$ be a MW-homotopy sheaf. For any smooth scheme X , we have a canonical isomorphism

$$b_X : M \rightarrow \tilde{\Gamma}_* \tilde{\Gamma}^*(M)(X)$$

which is compatible with (smooth) pullbacks and the GW-action. Since push-forward p_* of a finite map $p : Y \rightarrow X$ is locally given by the multiplication by the correspondence $\alpha(p, \psi_l, L_l)$ of 5.1.12, we see that b commutes with p_* . Thus b commutes with the multiplication by any cycle $\alpha \in \widetilde{\text{CH}}_T^{d_Y}(X \times Y, \omega_{Y/k})$ (where X, Y are two smooth schemes and $T \in X \times Y$ is an admissible subset) thanks to the identity

$$\alpha^*(\beta) = (p_X)_*(\alpha \cdot p_Y^*(\beta))$$

where $p_X : T \rightarrow X \times Y$ and $p_Y : X \times Y \rightarrow Y$ are the canonical morphisms.

6. Applications

6.1. Infinite suspensions of homotopy sheaves

In order to fix notations, we recall that we have the following commutative diagram of categories :

$$\begin{array}{ccccc}
\mathbf{H}_\bullet(k) & \xrightleftharpoons[\Omega_{S^1}^\infty]{\Sigma_{S^1}^\infty} & \mathbf{SH}^{S^1}(k) & \xrightleftharpoons[\Omega_{\mathbb{G}_m}^\infty]{\Sigma_{\mathbb{G}_m}^\infty} & \mathbf{SH}(k) \\
& & \uparrow K \downarrow N & & \uparrow K \downarrow N \\
& & \mathbf{D}_{\mathbb{A}^1}^{\text{eff}}(k) & \xrightleftharpoons[\Omega^\infty]{\Sigma^\infty} & \mathbf{D}_{\mathbb{A}^1}(k)
\end{array}$$

where $\mathbf{H}_\bullet(k)$ is the pointed unstable homotopy category, $\mathbf{SH}^{S^1}(k)$ (resp. $\mathbf{SH}(k)$) is the category obtain after S^1 -stabilization (resp. \mathbb{P}^1 -stabilization) and $\mathbf{D}_{\mathbb{A}^1}^{\text{eff}}(k)$ (resp. $\mathbf{D}_{\mathbb{A}^1}(k)$) the (resp. stable) effective \mathbb{A}^1 -derived category (see (Cisinski and Déglise, 2019, §5)); all of these triangulated categories are equipped with Morel's homotopy t-structure.

Since (N, K) is an equivalence that respects the t-structure, we have an equivalence (of additive symmetric monoidal categories) $\mathbf{D}_{\mathbb{A}^1}(k)^\heartsuit \simeq \mathbf{SH}(k)^\heartsuit$ (see (Déglise, 2018, §1.2.4)); since $(\Sigma^\infty, \Omega^\infty)$ is an adjunction that respects the t-structure, we also have an adjunction on the respective hearts (see also *ibid.* §4):

$$\mathbf{HI}(k) \xrightleftharpoons[\omega^\infty]{\sigma^\infty} \mathbf{HM}(k)$$

where we recall that $\mathbf{HM}(k)$ is the category of homotopy modules.

6.1.1. We also have the following commutative diagram of categories

$$\begin{array}{ccc}
\mathbf{D}_{\mathbb{A}^1}^{\text{eff}}(k) & \xrightarrow{\Sigma^\infty} & \mathbf{D}_{\mathbb{A}^1}(k) \\
\downarrow & & \downarrow \\
\widetilde{\mathbf{DM}}^{\text{eff}}(k) & \xrightarrow{\Sigma_{\text{MW}}^\infty} & \widetilde{\mathbf{DM}}(k)
\end{array}$$

where $\widetilde{\mathbf{DM}}^{\text{eff}}(k)$ and $\widetilde{\mathbf{DM}}(k)$ are the categories of (effective) MW-motivic complexes (see (Bachmann et al., 2020, Chapter 3)). Looking at the respective hearts, we thus obtain the following commutative diagram

$$\begin{array}{ccc}
\mathbf{HI}(k) & \xrightleftharpoons[\omega^\infty]{\sigma^\infty} & \mathbf{HM}(k) \\
\tilde{\gamma}_* \uparrow \downarrow \tilde{\gamma}^* & & \gamma_* \uparrow \downarrow \gamma^* \\
\mathbf{HI}^{\text{MW}}(k) & \xrightleftharpoons[\omega_{\text{MW}}^\infty]{\sigma_{\text{MW}}^\infty} & \mathbf{HM}^{\text{MW}}(k)
\end{array}$$

where we recall that $\mathbf{HM}^{\text{MW}}(k)$ denotes the category of Milnor-Witt homotopy modules, i.e. the category of pairs (M_*, ω_*) where M_* is a \mathbb{Z} -graded homotopy invariant sheaf with MW-transfers and $\omega_i : M_i \rightarrow (M_{i+1})_{-1}$ are isomorphisms of sheaves with MW-transfers.

Finally, we recall the two following well-known theorems in order to motivate Theorem 6.1.6.

Theorem 6.1.2. *With the previous notations, the adjunction*

$$\mathbf{HM}(k) \begin{array}{c} \xrightarrow{\gamma^*} \\ \xleftarrow{\gamma_*} \end{array} \mathbf{HM}^{\text{MW}}(k).$$

is an equivalence of categories.

PROOF. See (Feld, 2021, Theorem 5.9).

Theorem 6.1.3. *With the previous notations, the functor $\sigma_{\text{MW}}^\infty : \mathbf{HI}^{\text{MW}}(k) \rightarrow \mathbf{HM}^{\text{MW}}(k)$ is fully faithful.*

PROOF. According to (Bachmann et al., 2020, Chapter 3, Cor. 3.3.9), the functor $\Sigma_{\text{MW}}^\infty$ is fully faithful (because our base field k is infinite). This fact implies that $\sigma_{\text{MW}}^\infty$ is also fully faithful⁷. Indeed, for any sheaf $M \in \mathbf{HI}^{\text{MW}}(k)$, we have $\omega_{\text{MW}}^\infty(M) = \Omega_{\text{MW}}^\infty(M) = H_0\Omega_{\text{MW}}^\infty(M)$ and $\sigma_{\text{MW}}^\infty(M) = \tau_{\leq 0}\Sigma_{\text{MW}}^\infty(M) = H_0\Sigma_{\text{MW}}^\infty(M)$ hence the arrow

$$\text{Id} \rightarrow \omega_{\text{MW}}^\infty \sigma_{\text{MW}}^\infty = H_0(\text{Id} \rightarrow \Omega_{\text{MW}}^\infty \Sigma_{\text{MW}}^\infty)$$

is an isomorphism if $\Sigma_{\text{MW}}^\infty$ is fully faithful.

Remark 6.1.4. A consequence of Theorem 6.1.3 is that, if $M \in \mathbf{HI}^{\text{MW}}(k)$ is a sheaf with MW-transfers, then for any natural number n , there exists a sheaf with MW-transfers $N \in \mathbf{HI}^{\text{MW}}(k)$ such that $M \simeq N_{-n}$.

6.1.5. Theorem 6.1.2 and Theorem 6.1.3 suggest that similar results should hold for the functor $\tilde{\gamma}_* : \mathbf{HI}^{\text{MW}}(k) \rightarrow \mathbf{HI}(k)$ that forgets MW-transfers. This functor is clearly faithful and conservative but cannot be full according to the following counterexample due to Bachmann:

⁷Note that the converse is (in general) not true for categories with a t-structure.

Consider the constant sheaf \mathbb{Z} , which admits MW-transfers. For any MW-homotopy sheaf F , the set of maps $\mathrm{Hom}_{\mathbf{HI}^{\mathrm{MW}}}(\mathbb{Z}, F)$ injects into the subset of $F(k)$ given by the annihilator of the fundamental ideal \mathbf{I} of $\mathrm{GW}(k)$ acting on $F(k)$ (since $\mathbb{Z} = \mathrm{GW} / \mathbf{I}$ and maps with MW-transfers from GW to F are given by $F(k)$). On the other hand, the set of maps $\mathrm{Hom}_{\mathbf{HI}}(\mathbb{Z}, F)$ is given by all of $F(k)$.

However, we can still characterize its essential image thanks to the following theorem.

Theorem 6.1.6. *Let $M \in \mathbf{HI}(k)$ be a homotopy sheaf. The following assertions are equivalent:*

- (i) *There exists $M' \in \mathbf{HI}(k)$ satisfying Conjecture 4.1.13 and such that $M \simeq M'_{-1}$.*
- (ii) *There exists a structure of generalized transfers on M .*
- (iii) *There exists a structure of MW-transfers on M .*
- (iv) *There exists $M'' \in \mathbf{HI}(k)$ such that $M \simeq M''_{-2}$.*

PROOF. (i) \Rightarrow (ii) See Theorem 4.1.21.

(ii) \Rightarrow (iii) See Theorem 5.2.5.

(iii) \Rightarrow (iv) Assume $M \in \mathbf{HI}^{\mathrm{MW}}(k)$. Put $M_* = \sigma_{\mathrm{MW}}^\infty(M) \in \mathbf{HM}^{\mathrm{MW}}(k)$ so that we have $\omega_{\mathrm{MW}}^\infty(M_*) = M_0 \simeq (M_2)_{-2}$. Since $\sigma_{\mathrm{MW}}^\infty$ is fully faithful, the map $M \rightarrow \omega_{\mathrm{MW}}^\infty \sigma_{\mathrm{MW}}^\infty(M)$ is an isomorphism thus $M \simeq M''_{-2}$ with $M'' = M_2$.

(iv) \Rightarrow (i) Straightforward.

Remark 6.1.7. Keeping the previous notations, we remark that the equivalence (i) \Leftrightarrow (ii) was conjectured by Morel in (Morel, 2011, Remark 5.10).

6.2. Towards conservativity of \mathbb{G}_m -stabilization

We end with a discussion about a conjecture introduced in Bachmann and Yakerson (2020). In the classical theory of topological spaces, the functor $\mathrm{Spc}_* \rightarrow \mathfrak{D}(\mathcal{A}b)$, sending a space to its singular chain complex, is conservative on (at least) simply connected spaces. We would like to study a similar question in the motivic context: up to which extent is the functor

$$\Sigma_{\mathbb{G}_m}^\infty : \mathbf{SH}^{S^1}(k) \rightarrow \mathbf{SH}(k)$$

conservative? The conjecture of Bachmann and Yakerson relied on the hope that this is true after \mathbb{G}_m -suspension. Precisely, for any natural number n , denote by $\mathbf{SH}^{S^1}(k)(n)$ the localizing subcategory of $\mathbf{SH}^{S^1}(k)$ generated by $\Sigma_{\mathbb{G}_m}^n \Sigma_{S^1}^\infty X_+$ where $X \in \mathbf{Sm}_k$. The fact that the functor $\Sigma_{\mathbb{G}_m}^\infty : \mathbf{SH}^{S^1}(k)(1) \rightarrow \mathbf{SH}(k)$ is conservative on bounded below objects reduces to proving the following statement (see (Bachmann and Yakerson, 2020, Conjecture 1.1)).

Conjecture 6.2.1 (Bachmann-Yakerson). If $d \geq 1$ is a natural number, then the canonical functor

$$\Sigma_{\mathbb{G}_m}^{\infty-d\heartsuit} : \mathbf{SH}^{S^1}(k)(d)^\heartsuit \rightarrow \mathbf{SH}(k)^{eff\heartsuit}$$

is an equivalence of abelian categories.

Recall that $\mathbf{SH}(k)^{eff}$ denotes the localizing subcategory generated by the image of $\mathbf{SH}^{S^1}(k)$ in $\mathbf{SH}(k)$ under $\Sigma_{\mathbb{G}_m}^\infty$ and the hearts are taken with respect to homotopy t-structures on these categories. As a reformulation of the conjecture, we remark that the category $\mathbf{SH}^{S^1}(k)^\heartsuit$ is equivalent to the category of homotopy sheaves $\mathbf{HI}(k)$ and that the category $\mathbf{SH}(k)^{eff\heartsuit}$ is equivalent to the category $\mathbf{HI}^{fr}(k)$ of homotopy sheaves with *framed transfers* (see (Bachmann and Yakerson, 2020, Theorem 5.14)). We have the following theorem.

Theorem 6.2.2. *Let $d > 0$ be a natural number. If $d > 1$ or Conjecture 4.1.13 holds, then for any homotopy sheaf $M \in \mathbf{HI}(k)$, the Bass-Tate transfers on M_{-d} extend to framed transfers and the canonical functor*

$$\Sigma_{\mathbb{G}_m}^{\infty-d\heartsuit} : \mathbf{SH}^{S^1}(k)(d)^\heartsuit \rightarrow \mathbf{SH}(k)^{eff\heartsuit}$$

is an equivalence of abelian categories.

PROOF. Let $M \in \mathbf{HI}(k)$ be a homotopy sheaf. If $d > 1$ (resp. if Conjecture 4.1.13 holds), then M_{-d} (resp. M_{-1}) has a structure of generalized transfers hence of Milnor-Witt transfers according to Subsection 5.2. Thus it has a structure of framed transfers (see (Bachmann et al., 2020, Chapter 3, §2)) and the first result holds. The second one is (Bachmann, 2020, Theorem 4.5).

As an application of our theorem 4.1.21, we obtain:

Corollary 6.2.3. *Let $d > 0$ be a natural number. The Bachmann-Yakerson conjecture holds (integrally) for $d = 2$ and rationally for $d = 1$: namely, the canonical functor*

$$\mathbf{SH}^{S^1}(k)(2) \rightarrow \mathbf{SH}(k)$$

is conservative on bounded below objects, the canonical functor

$$\mathbf{SH}^{S^1}(k)(1) \rightarrow \mathbf{SH}(k)$$

is conservative on rational bounded below objects, and the canonical functor

$$\mathbf{HI}(k, \mathbb{Q})(1) \rightarrow \mathbf{HI}^{\mathrm{fr}}(k, \mathbb{Q})$$

is an equivalence of abelian categories.

Moreover, let \mathcal{X} be a pointed motivic space. Then the canonical map

$$\pi_0 \Omega_{\mathbb{P}^1}^d \Sigma_{\mathbb{P}^1}^d \mathcal{X} \rightarrow \pi_0 \Omega_{\mathbb{P}^1}^{d+1} \Sigma_{\mathbb{P}^1}^{d+1} \mathcal{X}$$

is an isomorphism for $d = 2$.

PROOF. The last result follows as in the proof of (Bachmann, 2020, Theorem 1.1).

Moreover, we can solve (integrally) a question left open after the work of Bachmann et al. (2020) and Garkusha and Panin (2021):

Corollary 6.2.4. *The category of homotopy sheaves with generalized transfers, the category of MW-homotopy sheaves and the category of homotopy sheaves with framed transfers are equivalent:*

$$\mathbf{HI}^{\mathrm{gtr}}(k) \simeq \mathbf{HI}^{\mathrm{MW}}(k) \simeq \mathbf{HI}^{\mathrm{fr}}(k).$$

PROOF. The first equivalence is our Theorem 5.2.5; the second one is due⁸ to Bachmann (combine (Bachmann, 2021, Proposition 29) and (Bachmann and Yakerson, 2020, Theorem 5.14)).

Remark 6.2.5. We end with a remark on the characteristic of the base field k . Indeed, we have assumed that $\mathrm{char}(k) \neq 2$ but we believe that the assumption could be lifted. For that, the foundations of the theory of Milnor-Witt correspondence as developed in Bachmann et al. (2020) should be extended to the case of characteristic 2 (in particular, we would like to

⁸The equivalence between $\mathbf{HI}^{\mathrm{MW}}(k)$ and $\mathbf{HI}^{\mathrm{fr}}(k)$ also appears in (Ananyevskiy and Neshitov, 2019, Theorem 8.12).

prove the cancellation theorem for Milnor-Witt transfers (Fasel and Østvær, 2017, Theorem 4.0.1) in full generality).

If one is interested only in the applications in Section 6 for fields of characteristic 2, then one may also try to work exclusively with framed transfers since we know that the cancellation theorem is true for framed correspondences in characteristic 2 (see Ananyevskiy et al. (2021)).

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