Birational invariance of the Chow-Witt group of zero-cycles

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Abstract

We prove that the Chow-Witt group of zero-cycles is a birational invariant of smooth proper schemes over a base field.

Contents

Introduction

The notion of *Milnor-Witt cycle modules* is introduced by the author in [Fel20, Fel21b] over a perfect field k which, after slight changes, can be generalized to more general base schemes (see [BHP22] for the case of a regular base scheme, and [DFJ22] for any base schemes).

The main example of a Milnor-Witt cycle module is given by the Milnor-Witt K-theory $\underline{\mathrm{K}}^{MW}$ (see [BCD$^+$20, Fel21c, Fel21d, Fel21a].

To any MW-cycle module M and any k-scheme X equipped with a line bundle l_X, one can associated a Rost-Schmid complex $C_*(X, M, l_X)$ whose homology groups are called the called *the Chow-Witt groups with coefficient in* M. In particular, if $M = \underline{\mathrm{K}}^{MW}$, one recovers the Chow-Witt groups $\widetilde{\mathrm{CH}}_*(X, l)$ (see [Fas20]) which are, in some sense, a *quadratic refinement* of the classical Chow group $\mathrm{CH}_*(X)$.

A well-known consequence of intersection theory is that the Chow group $\mathrm{CH}_0(X)$ is a birational invariant. Indeed, a partial result was proved in [CC79]. The case of an algebraically closed base field can be found in [Ful98, Example 16.1.11]. The general case follows verbatim from the proof of Fulton, according to [vDdB16]. It is also a consequence of Theorem 2.2.12.

A natural question is wether or not the birational invariance holds true for the Chow-Witt group and, more generally, of the Chow-Witt groups with coefficients in a Milnor-Witt cycle module). It is easy to see that the Chow-Witt group in **cohomological** degree zero $\widetilde{\mathrm{CH}}^0$ is a birational invariant for smooth proper k-scheme (see [Fel21b, Theorem 5.6]). In homological degree zero, the question is more complex.

Following ideas of Merkurjev [KM13], we prove that the Chow-Witt group of zero-cycles is a birational invariant for smooth proper schemes. More generally, we have:

Theorem 1 (see Theorem 2.2.12)**.** The group $A_0(X, M)$ is a birational invariant of the smooth proper scheme X.

In particular, the Chow-Witt group of zero-cycles $\widetilde{\mathrm{CH}}_0(X)$ is a birational invariant of the smooth proper scheme X.

Outline of the paper

In Section 1, we explain how to build a special type of Milnor-Witt cycle module from a fix MW-module. Moreover, we define a cup product for oriented schemes.

In Section 2, we prove that the two previous constructions are compatible with each other in some sense. This allows us to define a composition of *Milnor-Witt rational correspondences* and construct an associated pushforward map. Finally, we apply these results to prove that Chow-Witt group of zero-cycles is a birational invariant for smooth proper schemes.

In Appendix A, we recall the basic definitions of (cohomological) Milnor-Witt cycle modules along with the basic maps (pushforward, pullback, etc.). We then define the new class of *oriented* schemes.

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Notations and conventions

In this paper, schemes are noetherian and finite dimensional. We fix a base field[1] k and put $S = \operatorname{Spec} k$, and we fix a base ring of coefficients R. If not stated otherwise, all schemes and morphisms of schemes are defined over S. A *point* (resp. *trait*, *singular trait*) of S will be a morphism of schemes $\operatorname{Spec}(k) \to S$ essentially of finite type and such

Conventions: a morphism $f : X \to S$ (sometime denoted by X/S) is:

- essentially of finite type if f is the projective limit of a cofiltered system $(f_i)_{i \in I}$ of morphisms of finite type with affine and étale transition maps
- lci if it is smoothable and a local complete intersection (*i.e.* admits a global factorization $f = p \circ i$, p smooth and i a regular closed immersion);
- essentially lci if it is a limit of lci morphisms with étale transition maps.

Let X/S be a scheme essentially of finite type. We put $X_{(p)}$ the set of p-dimensional points of X.

A point x of S is a map $x : \operatorname{Spec}(E) \to S$ essentially of finite type and such E is a field. We also say that E is a field over S.

Given a morphism of schemes $f : Y \to X$, we let L_f be its cotangent complex, an object of $\mathrm{D}^b_{\mathrm{coh}}(Y)$, and when the latter is perfect (e.g. if f is essentially lci), we let τ_f be its associated virtual vector bundle over Y, and by ω_f the determinant of τ_f.

If not stated otherwise, M is a (cohomological) Milnor-Witt cycle module, X is an S-scheme, l is a line bundle over X, and p, q are integers.

1 Main constructions

1.1 The relative perverse homology

We follow [Ros96, §7]. In this section, we show that new Milnor-Witt cycle modules can be obtained from the Chow groups of the fibers of a morphism.

[1] Many results of the present paper are in fact true over a more general base scheme.

1.1.1. Let $\rho : Q \to S$ be a morphism of finite type and let M be a cohomological MW-cycle module over Q. Fix l a line bundle over Q. For any field F over S, denote by $Q_F = Q \times_B \operatorname{Spec} F$. We define an object function $A_p[\rho, M, l]$ on $\mathbf{F}(S)$ by

$$A_p[\rho, M, l] = \bigoplus_{q \in \mathbf{Z}} A_p[\rho, M_q, l]$$

where

$$A_p[\rho, M_q, l](F) = A_p(Q_F, M_q, \omega^\vee_{Q_F/Q} \otimes l).$$

Our aim is to show that $A_p[\rho, M, l]$ is in a natural way a Milnor-Witt cycle module over S.

1.1.2. All the properties of Milnor-Witt cycle modules except axiom (C) hold already on complex level, i.e. for the groups $C_p(Q_F, M)$. Indeed, we denote by $\widehat{M}$ the object function on $\mathbf{F}(B)$ defined by

$$\widehat{M}(F) = C_p(Q_F, M, \omega^\vee_{Q_F/Q} \otimes l) = \bigoplus_{q \in \mathbf{Z}} C_p(Q_F, M_q, \omega^\vee_{Q_F/Q}).$$

We first describe its data as a Milnor-Witt cycle premodule. These will be denoted by $\widehat{\operatorname{res}}_{F/E}$, $\widehat{\operatorname{cores}}_{F/E}$, etc. in order to distinguish them from the data $\operatorname{res}_{F/E}$, $\operatorname{cores}_{F/E}$, etc. of M.

For a morphism of fields $\phi : E \to F$, let $\overline{\phi} : Q_F \to Q_E$ be the induced map.

1. DATA D1 Define

$$\widehat{\operatorname{res}}_{F/E} := \phi^! : C_p(Q_E, M_q, \omega^\vee_{Q_E/Q}) \to C_p(Q_F, M_q, \omega^\vee_{Q_F/Q}).$$

2. DATA D2 Assume ϕ finite. Define

$$\widehat{\operatorname{cores}}_{F/E} := \phi_* : C_p(Q_F, M_q, \mathcal{O}_{Q_F}) \to C_p(Q_E, M_q, \mathcal{O}_{Q_E}).$$

3. DATA D3 Simply take the $\underline{\mathrm{K}}^{MW}$-module structure on $C_p(Q_F, M)$ described in [DFJ22, §1.4 and §5.4].

4. DATA D4 Denote by $\widetilde{Q}_v = Q \times_S \operatorname{Spec} \mathcal{O}_v$, the generic fiber Q_F and the special fiber $Q_{\kappa(v)}$. Define

$$\widehat{\partial}_v : C_p(Q_F, M_q) \to C_{p-1}(Q_{\kappa(v)}, M_q)$$

by $(\widehat{\partial}_v)^x_y = \partial^x_y$ with ∂^x_y as in [DFJ22, §5.3.13] with respect to the scheme $\widetilde{Q_v}$.

Theorem 1.1.3. *Keeping the previous notations, the object functor $\widehat{M}$ along with these data form a Milnor-Witt cycle premodule over S.*

Proof. All the required properties follow from the rules and axioms for M and from the functorial properties studied in [DFJ22, §1.4 and §5.4]. $\square$

1.1.4. Now, we want to relate the differentials for the MW-cycle premodule $\widehat{M}$ to the differentials for the MW-cycle module M.

Let $X \to S$ be a scheme over S and let $\widetilde{X} = Q \times_S X$. Then for x, y in X, there is a map

$$\widehat{\partial_y^x} : \widehat{M}(x) \to \widehat{M}(y)$$

as in [DFJ22, §5.3.13]. By definition, this is a map

$$\widehat{\partial}_y^x : C_p(Q_{\kappa(x)}, M) \to C_p(Q_{\kappa(y)}, M)$$

between cycle groups with coefficients in M.

Proposition 1.1.5. *Let $\widetilde{x}, \widetilde{y}$ in $\widetilde{X}$ be points lying over $x, y \in X$, respectively, and assume that $\widetilde{x} \in (Q_{\kappa(x)})_{(q)}$ and $\widetilde{y} \in (Q_{\kappa(y)})_{(q)}$. Denote by $(\widehat{\partial_y^x})_{\widetilde{y}}^{\widetilde{x}}$ the component of $\widehat{\partial_y^x}$ with respect to $\widetilde{x}$ and $\widetilde{y}$. Then*

$$(\widehat{\partial}_y^x)_{\widetilde{y}}^{\widetilde{x}} = \partial_{\widetilde{y}}^{\widetilde{x}} : M_q(\widetilde{x}, \omega_{\widetilde{x}/S}) \to M_{q-1}(\widetilde{y}, \omega_{\widetilde{y}/S}).$$

Proof. We may assume $\widetilde{y} \in \overline{\{\widetilde{x}\}}^{(1)}$, since otherwise both sides are trivial. The dimension inequality [Mat80, p. 85] shows then $y \in \overline{\{x\}}^{(1)}$. Let v run through the valuations of $\kappa(x)$ with center y in X. Moreover, let w run through the valuations on $\kappa(\widetilde{x})$ with center $\widetilde{y}$ in $\widetilde{X}$. The restriction of any w to $\kappa(x)$ is one of the valuations v. Let $\widetilde{w} \in Q_{\kappa(v)}$ be the center of w in $\widetilde{X} \times_X \operatorname{Spec} \mathcal{O}_v$. Now the claim follows from

$$\begin{array}{rcl} (\widehat{\partial}_y^x)_{\widetilde{y}}^{\widetilde{x}} & = & (\sum_v \widehat{\operatorname{cores}}_{\kappa(v)/\kappa(y)} \circ \widehat{\partial}_v)_{\widetilde{y}}^{\widetilde{x}} \\ & = & \sum_v \sum_{w|v} (\widehat{\operatorname{cores}}_{\kappa(v)/\kappa(y)})_{\widetilde{y}}^{\widetilde{w}} \circ (\widehat{\partial}_v)_{\widetilde{w}}^{\widetilde{x}} \\ & = & \sum_v \sum_{w|v} \operatorname{cores}_{\kappa(\widetilde{w}/\kappa(\widetilde{y})} \circ \operatorname{cores}_{\kappa(w)|\kappa(\widetilde{w})} \circ \partial_w \\ & = & \sum_w \operatorname{cores}_{\kappa(w)/\kappa(\widetilde{y})} \circ \partial_w \\ & = & \partial_{\widetilde{y}}^{\widetilde{x}}. \end{array}$$

$\square$

It follows from [DFJ22, Proposition 1.4.6] that the data of the MW-cycle premodule $\widehat{M}$ commute with the differentials of the complex $C_*(Q_F, M)$. Passing to homology, we obtain data D1-D4 for the object function $A_q[\rho, M]$.

Theorem 1.1.6. *Keeping the previous notations, the object function $A_p[\rho, M]$ together with these data is a Milnor-Witt cycle module over S.*

Proof. The rules for the data of the MW-cycle premodule $A_p[\rho, M]$ are immediate from the rules for $\widehat{M}$. Moreover, axiom (FD) for M and Proposition 1.1.5 show that (FD) holds for $\widehat{M}$ and thus for $A_p[\rho, M]$. It remains to verify axiom (C).

Consider the map

$$C_p(Q_{\kappa(\xi)}) \xrightarrow{\delta} C_{p-1}(Q_{\kappa(\xi)}) \oplus \bigoplus_{x\in X^{(1)}} C_p(Q_{\kappa(x)}) \oplus C_{p+1}(Q_{\kappa(x_0)}) \xrightarrow{\delta} C_p(Q_{\kappa(x_0)})$$

defined by $\delta^z_y = \partial^z_y$ with ∂^z_y as in [DFJ22, §5.3.13] with respect to the scheme $Q \times_B X$ (we have shortened the notation by omitting M).

By Proposition 1.1.5, we are reduced to show $\delta \circ \delta = 0$. It suffices to check that $(\delta \circ \delta)^z_y = 0$ for $z \in (Q_{\kappa(\xi)})_{(q)}$ and $y \in (Q_{\kappa(x_0)})_{(q)}$ with $y \in \overline{\{z\}}^{(2)}$ (here $\overline{\{z\}}$ is the closure of z in $\widetilde{X}$). The dimension inequality [Mat80, p. 85] shows

$$Z^{(1)} \subset (Q_{\kappa(\xi)})_{(q-1)} \cup \bigcup_x (Q_{\kappa(x)})_{(q)} \cup (Q_{\kappa(x_0)})_{(q+1)}$$

with $Z = \overline{\{z\}}_{(y)}$. We are done by axiom (C) for M. □

Definition 1.1.7. Keeping the previous notations, the Milnor-Witt cycle module $A_p[\rho, M]$ is called the *p-th relative perverse homology* of M with respect to ρ.

Remark 1.1.8. One should also obtain the results present in [Ros96, §8]. In particular, the MW-cycle module $A_q[\rho, M]$ could be used to give another proof of the homotopy invariance of the Rost-Schmid complex.

1.2 The cup product

1.2.1. We follow ideas of Merkurjev [Mer03]. We work over a base field k. We fix M a Milnor-Witt cycle module over k.

1.2.2. Let $M \times N \to P$ be a bilinear pairing of MW-cycle modules over k. Let X, Y and Z be smooth schemes over k with Y irreducible smooth and proper. Denote by $\Delta : Y \to Y \times Y$ the diagonal map. Let q be an integer and l_X (resp. l_Y, l'_Y, and l'_Z) a line bundle over X (resp. Y, Y, and Z). Assume that $\omega_{Y/k} \otimes l_Y \otimes l'_Y \simeq \mathcal{O}_Y$.

We have a $\cup$-product

$$\begin{aligned}\cup : A_r(X \times Y, M_s, l_X \otimes l_Y) \otimes A_p(Y \times Z, N_q, l'_Y \otimes l'_Z) \to \\ A_{r+p-d_Y}(X \times Z, P_{s+q+d_Y}, l_X \otimes l_Z)\end{aligned}$$

defined as the composition

$$
\begin{array}{c}
A_r(X\times Y, M_s, l_X\otimes l_Y)\otimes A_p(Y\times Z, N_q, l'_Y\otimes l'_Z)\\
\Big\downarrow{\scriptstyle\times}\\
A_{r+p}(X\times Y\times Y\times Z, P_{q+s}, l_X\otimes l_Y\otimes l'_Y\otimes l'_Z)\\
\Big\downarrow{\scriptstyle(\mathrm{Id}_X\otimes\Delta\otimes\mathrm{Id}_Z)^*}\\
A_{r+p-d_Y}(X\times Y\times Z, P_{s+q+d_Y}, l_X\otimes\omega_{Y/k}\otimes l_Y\otimes l'_Y\otimes l'_Z)\\
\Big\downarrow{\scriptstyle\simeq}\\
A_{r+p-d_Y}(X\times Y\times Z, P_{s+q+d_Y}, l_X\otimes l'_Z)\\
\Big\downarrow{\scriptstyle\pi_{XZ*}}\\
A_{r+p-d_Y}(X\times Z, P_{s+q+d_Y}, l_X\otimes l'_Z)
\end{array}
$$

where $\times$ is the cross product (see [Fel21b, Section 10]), $\Delta : Y \to Y\times Y$ is the diagonal embedding and $\pi_{XZ} : X\times Y\times Z \to X\times Z$ is the projection. The pushforward p_{XZ*} is well-defined because Y is smooth and proper.

1.2.3. In particular, taking $N = M = P = \underline{\mathrm{K}}^{MW}$, $l_X = \omega^\vee_{X/k}$, $l_Y = \mathcal{O}_Y$, $l'_Y = \omega^\vee_{Y/k}$, $l'_Z = \mathcal{O}_Z$, $r = -s$ and $p = -q$, we have the product

$$\cup : \widetilde{\mathrm{CH}}_r(X\times Y, \omega^\vee_{X/S})\otimes\widetilde{\mathrm{CH}}_p(Y\times Z, \omega^\vee_{Y/S})\to\widetilde{\mathrm{CH}}_{r+p-d_Y}(X\times Z, \omega^\vee_{X/S})$$

which could be taken as the composition law for the category of Milnor-Witt integral correspondences $\widetilde{\mathbf{Cor}}$ with objects the smooth proper schemes over k and morphisms

$$\mathrm{Hom}_{\widetilde{\mathbf{Cor}}}(X,Y) = \bigoplus_i \widetilde{\mathrm{CH}}_{d_i}(X_i\times Y, \omega^\vee_{X/S}),$$

where X_i are irreducible (connected) components of X with $d_i = \dim X_i$.

2 Milnor-Witt rational correspondences

Let X be a smooth and proper k-scheme and l_X (resp. l_Y) a line bundle over X (resp. Y). There is a canonical map of complexes

$$\Theta_M : C_p(X\times Y, M_q, l_X\otimes l_Y)\to C_p(X, A_0[Y, M_q, l_Y], l_X),$$

that takes an elements in $M(z, \omega_z\otimes l_{X|z}\otimes l_{Y|z})$ for $z\in(X\times Y)_{(p)}$ to zero if dimension of the projection x of z in X is strictly less than p, and identically to itself otherwise. In the latter case, we consider z as a point of dimension 0 in $Y_x := Y_{\kappa(x)}$ under the inclusion $Y_x\subset X\times Y$. Thus, $\Theta_{Y,M}$ "ignores" points in $X\times Y$ that lose dimension being projected to X.

We study various compatibility properties of Θ_M.

2.1 Cross products

Let $M \times N \to P$ be a bilinear pairing of MW-cycle modules over k. For a smooth scheme Y over k and l_Y a line bundle over Y, we can define a pairing

$$M \times A_0[Y, N, l_Y] \to A_0[Y, P, l_Y]$$

in an obvious way.

Lemma 2.1.1. *For X, Y, Z smooth k-schemes, and l_X (resp. l_Y, l_Y) a line bundle over X (resp. Y, Z), the following diagram is commutative:*

$$\begin{tikzcd}
C_p(X, M_q, l_X) \otimes C_r(Y \times Z, N_s, l_Y \otimes l_Z) \arrow[r, "\times"] \arrow[d, "\mathrm{Id} \times \Theta_N"] & C_{p+r}(X \times Y \times Z, P_{q+s}, l_X \otimes l_Y \otimes l_Z) \arrow[d, "\Theta_P"] \\
C_p(X, M_q, l_X], l_Y) \otimes C_r(Y A_0[Z, N_s, l_Z], l_Y) \arrow[r, "\times"] & C_{p+r}(X \times Y, A_0[Z, P_{q+s}, l_Z], l_X \otimes l_Y).
\end{tikzcd}$$

Proof. Let $x \in X_{(p)}$ and $\mu \in C_p(X, M_s, l_X)$. Consider the following commutative diagram

$$\begin{tikzcd}
C_r(Y \times Z, N_s, l_Y \otimes l_Z) \arrow[r, "\Theta_N"] \arrow[d, "\pi'^*_x"] & C_r(Y, A_0[Z, N_s, l_Z], l_Y) \arrow[d, "\pi^*_x"] \\
C_r((Y \times Z)_x, N_s, l_Y \otimes l_Z) \arrow[r, "\Theta_N"] \arrow[d, "m'_\mu"] & C_p(Y_x, A_0[Z, N_s, l_Z], l_Y) \arrow[d, "m_\mu"] \\
C_{p+r}((Y \times Z)_x, P_{q+s}, l_X \otimes l_Y \otimes l_Z) \arrow[r, "\Theta_P"] \arrow[d, "i'_{x,*}"] & C_{p+r}(Y_x, A_0[Z, P_{q+s}, l_Z], l_X \otimes l_Y) \arrow[d, "i_{x,*}"] \\
C_{p+r}(X \times Y \times Z, P_{q+s}, l_X \otimes l_Y \otimes l_Z) \arrow[r, "\Theta_P"] & C_{p+r}(X \times Y, A_0[Z, P_{q+s}, l_Z], l_X \otimes l_Y)
\end{tikzcd}$$

where $\pi_x : Y_x \to Y$ and $\pi'_x : (X \times Y)_x \to X \times Y$ are the natural projections, m_μ and m'_μ are the multiplications by μ, and $i_x : Y_x \to X \times Y$ and $i'_x : (Y \times Z)_z \to X \times Y \times Z$ are the inclusions. By the definition of the cross product, the compositions in the two rows of the diagram are the multiplications by μ.

□

2.1.2. PULLBACK MAPS Let $f : Z \to X$ be a regular closed embedding of smooth schemes of dimension s and l a line bundle over X. We denote by $N_{X/Z}$ the normal bundle over Z. For an smooth scheme Y, the closed embedding

$$f' = f \times \mathrm{Id}_Y : Z \times Y \to X \times Y$$

is also regular and the normal bundle $N_{X \times Y/Z \times Y}$ is isomorphic to $N_{X/Z} \times Y$.

Lemma 2.1.3. *The following diagram is commutative:*

$$\begin{tikzcd} A_p(X\times Y, M_q, l) \arrow[r, "f'^*"] \arrow[d, "\Theta_M"] & A_{p+s}(Z\times Y, M_{q-s}, l\otimes\omega_f^\vee) \arrow[d, "\Theta_M"] \\ A_p(X, A_0[Y, M_q], l) \arrow[r, "f^*"] & A_{p+s}(Z, A_0[Y, M_{q-s}], l\otimes\omega_f^\vee). \end{tikzcd}$$

Proof. Let $\pi_X : \mathbf{G}_m \times X \to X$ and $\pi'_X : \mathbf{G}_m \times X \times Y \to X \times Y$ be the natural projections. The following diagram

$$\begin{tikzcd} C_p(X\times Y, M_q, l) \arrow[r, "(\pi'_X)^*"] \arrow[d, "\Theta_M"] & C_{p+1}(\mathbf{G}_m\times X\times Y, M_{q-1}, l) \arrow[d, "\Theta_M"] \\ C_p(X, A_0[Y, M_q], l) \arrow[r, "\pi_X^*"] & C_{q+1}(\mathbf{G}_m\times X, A_0[Y, M_{q-1}], l) \end{tikzcd}$$

is commutative.

Let t be the coordinate function on $\mathbf{G}_m$. The map Θ_M commutes with the multiplication by t, i.e. the following diagram

$$\begin{tikzcd} C_p(\mathbf{G}_m\times X\times Y, M_q, l) \arrow[r, "{[t]}"] \arrow[d, "\Theta_M"] & C_p(\mathbf{G}_m\times X\times Y, M_{q+1}, l) \arrow[d, "\Theta_M"] \\ C_p(\mathbf{G}_m\times X, A_0[Y, M_q], l) \arrow[r, "{[t]}"] & C_p(\mathbf{G}_m\times X, A_0[Y, M_{q+1}], l) \end{tikzcd}$$

is commutative.

Let $D = D(X, Z)$ be the deformation space of the embedding f (see e.g. [Ros96, §10]). There is a closed embedding $i : N_{X/Z} \to D$ with the open complement $j : \mathbf{G}_m \times X \to D$. Then $D' = D \times Y$ is the deformation space $D(X \times Y, Z \times Y)$ with the closed embedding

$$i' = i \times \mathrm{Id}_Y : N_{X\times Y/Z\times Y} \to D'$$

and the open complement $j' = j \times \mathrm{Id}_Y : \mathbf{G}_m \times X \times Y \to D'$.

The commutative diagram with exact rows

$$\begin{tikzcd}
\vdots \arrow[d] & \vdots \arrow[d] \\
C_p(N_{X/Z}\times Y, M_q, l) \arrow[r, "\Theta_M"] \arrow[d, "i'_*"] & C_p(N_{X/Z}, A_0[Y, M_q], l) \arrow[d, "i_*"] \\
C_p(D', M_q, l) \arrow[r, "\Theta_M"] \arrow[d, "j'^*"] & C_p(D, A_0[Y, M_q], l) \arrow[d, "j^*"] \\
C_p(\mathbf{G}_m \times X \times Y, M_q, l) \arrow[r, "\Theta_M"] \arrow[d] & C_p(\mathbf{G}_m \times X, A_0[Y, M_q], l) \arrow[d] \\
\vdots & \vdots
\end{tikzcd}$$

induces the commutative diagram

$$\begin{tikzcd}
C_p(\mathbf{G}_m \times X \times Y, M_q, l) \arrow[r, "\partial"] \arrow[d, "\Theta_M"] & C_{p-1}(N_{X/Z}\times Y, M_q, l) \arrow[d, "\Theta_M"] \\
C_p(\mathbf{G}_m \times X, A_0[Y, M_q], l) \arrow[r, "\partial"] & C_p(N_{X/Z}, A_0[Y, M_q], l).
\end{tikzcd}$$

Finally, we also have the commutative diagram

$$\begin{tikzcd}
C_p(Z\times Y, M_q, l\otimes \omega_f^\vee) \arrow[r, "\pi^*"] \arrow[d, "\Theta_M"] & C_{p+s}(N_{X/Z}\times Y, M_{q-s}, l) \arrow[d, "\Theta_M"] \\
C_p(Z, A_0[Y, M_q], l\otimes \omega_f^\vee) \arrow[r, "\pi'^*"] & C_{p+s}(N_{X/Z}, A_0[Y, M_{q-s}], l)
\end{tikzcd}$$

where $\pi : N_{X/Z} \to Z$ is the canonical projection and s its relative dimension (π is a quasi-isomorphism by homotopy invariance). By the definition of the pullback map (see [Fel20, Section 7]), the result follows from the composition of the previous commutative square. $\square$

Remark 2.1.4. The previous lemma could be stated at the level of complexes with the use of Rost's coordinations or by using the homotopy complex defined in [DFJ22, §2.2], but we do not need this generality.

2.1.5. PUSHFORWARD MAPS Let $f : X \to Z$ be a map of oriented smooth schemes (over k). and l a line bundle over Z. For an oriented smooth scheme Y, set

$$f' = f \times \mathrm{Id}_Y : X \times Y \to Z \times Y.$$

Lemma 2.1.6. *The following diagram*

$$
\begin{array}{ccc}
C_p(X \times Y, M_q, l) & \xrightarrow{f'_*} & C_p(Z \times Y, M_q, l) \\
\downarrow{\scriptstyle \Theta_M} & & \downarrow{\scriptstyle \Theta_M} \\
C_p(X, A_0[Y, M_q], l) & \xrightarrow{f_*} & C_p(Z, A_0[Y, M_q], l).
\end{array}
$$

is commutative.

Proof. Let $u \in (X \times Y)_{(p)}$, $a \in M(\kappa(u), \omega_u \otimes l)$. Set $v = f'(u) \in Z \times Y$. If $\dim(v) < p$ then $(f'_*)_u(a) = 0$. In this case, the dimension of the projection y of u in Y is less than p and hence $(\Theta_M)_u(a) = 0$.

Assume that $\dim(v) = p$. Then $\kappa(u)/\kappa(v)$ is a finite field extension and

$$b = (f'_*)_u(a) = \mathrm{cores}_{\kappa(u)/\kappa(v)}(a) \in M(\kappa(v), \omega_v \otimes l).$$

If $\dim(y) < p$, then $(\Theta_M)_u(a) = 0$, and $\Theta_v(b) = 0$.

Assume that $\dim(y) = p$, then

$$(\Theta_M \circ f'_*)_u(a) = \mathrm{cores}_{\kappa(u)/\kappa(v)}(a) = b$$

considered as an element of $A_0[Y, M_q](\kappa(z), \omega_z \otimes l) = A_0(Y_z, M_q, l)$, where z is the image of v in Z. On the other hand,

$$(f_* \circ \Theta_M)_u(a) = \phi_*(a),$$

where $\phi : Y_x \to Y_z$ is the natural map (where x is the image of u in X) and is considered as an element of $A_0[Y, M_q](\kappa(z), \omega_z \otimes l)$. It remains to notice that

$$\phi_*(a) = \mathrm{cores}_{\kappa(u)/\kappa(v)}(a) = b.$$

□

2.2 Rational correspondences

Let Y and Z be smooth schemes over k. Assume Y irreducible and denote by d_Y the dimension of Y.By Lemma 2.1.1, for the pairing $M \times \underline{\mathrm{K}}^{MW} \to M$ and "$X = Y$" we have the commutative diagram

$$
\begin{array}{ccc}
A_0(Y, M_q) \otimes \widetilde{\mathrm{CH}}_{d_Y}(Y \times Z, \omega^\vee_{Y/k}) & \xrightarrow{\times} & A_{d_Y}(Y \times Y \times Z, M_{-d_Y+q}, \omega^\vee_{Y/k}) \\
\downarrow{\scriptstyle \mathrm{Id} \otimes \Theta_{\underline{\mathrm{K}}^{MW}}} & & \downarrow{\scriptstyle \Theta_M} \\
A_0(Y, M_q) \otimes A_{d_Y}(Y, A_0[Z, \underline{\mathrm{K}}^{MW}_{-d_Y}], \omega^\vee_{Y/k}) & \xrightarrow{\times} & A_{d_Y}(Y \times Y, A_0[Z, M_{-d_Y+q}], \omega^\vee_{Y/k}).
\end{array}
$$

Let $\Delta : Y \to Y \times Y$ be the diagonal embedding and $\Delta' = \Delta \otimes \mathrm{Id}_Z$. By Lemma 2.1.3, the following diagram

$$\begin{tikzcd}
A_{d_Y}(Y \times Y \times Z, M_{-d_Y+q}, \omega^\vee_{Y/k}) \arrow[r, "\Delta'^*"] \arrow[d, "\Theta_M"] & A_0(Y \times Z, M_q) \arrow[d, "\Theta_M"] \\
A_{d_Y}(Y \times Y, A_0[X, M_{-d_Y+q}], \omega^\vee_{Y/k}) \arrow[r, "\Delta^*"] & A_0(Y, A_0[Z, M_q])
\end{tikzcd}.$$

is commutative.

Finally, assume that the structure map $f : Y \to \operatorname{Spec} k$ is proper and denote by $f' = \mathrm{Id}_X \times f$. Lemma 2.1.6 implies that the following diagram

$$\begin{tikzcd}
A_0(Y \times Z, M_q) \arrow[r, "f'_*"] \arrow[d, "\Theta_M"] & A_0(Z, M_q) \arrow[d, equal] \\
A_0(Y, A_0[Z, M_q]) \arrow[r, "f_*"] & A_0(\operatorname{Spec} k, A_0[Z, M_q]).
\end{tikzcd}$$

is commutative.

Proposition 2.2.1. *Let Y and Z be smooth schemes over k, Y an irreducible smooth and proper, and M an MW-cycle module over k. Then the pairing*

$$\cup : A_0(Y, M_q) \otimes \widetilde{\mathrm{CH}}_{d_Y}(Y \times Z, \omega^\vee_{Y/k}) \to A_0(Z, M_q)$$

is trivial on all cycles in $\widetilde{\mathrm{CH}}_{d_Y}(Y \times Z, \omega^\vee_{Y/k})$ that are not dominant over Y. In other words, the $\cup$-product factors through a natural pairing

$$\cup : A_0(Y, M_q) \otimes \widetilde{\mathrm{CH}}_0(Z_{\kappa(Y)}, \omega^\vee_{Y/k}) \to A_0(Y, M_q)$$

Proof. This follows from composing all three diagrams and taking into account that

$$A_{d_Y}(Y, A_0[Z, \underline{\mathrm{K}}^{MW}_{-d_Y}], \omega^\vee_{Y/k}) = \widetilde{\mathrm{CH}}_0(Z_{\kappa(Y)}, \omega^\vee_{Y/k}).$$

□

2.2.2. Keeping the previous notations, for Z irreducible smooth scheme over k, the diagram

$$\begin{tikzcd}
A_{d_X}(X \times Y, M_q\omega^\vee_{X/k}) \otimes \widetilde{\mathrm{CH}}_{d_Y}(Y \times Z, \omega^\vee_{Y/k}) \arrow[r, "\cup"] \arrow[d] & A_{d_X}(X \times Z, M_q, \omega^\vee_{X/k}) \arrow[d] \\
A_0(Y_{\kappa(X)}, M_q, \omega^\vee_{X/k}) \otimes \widetilde{\mathrm{CH}}_{d_Y}(Y \times Z, \omega^\vee_{Y/k}) \arrow[r, "\cup"] \arrow[d] & A_0(Z_{\kappa(X)}, M_q, \omega^\vee_{X/k}) \arrow[d] \\
A_0(Y_{\kappa(X)}, M_q, \omega^\vee_{X/k}) \otimes \widetilde{\mathrm{CH}}_0(Z_{\kappa(Y)}, \omega^\vee_{Y/k}) \arrow[r, "\cup"] & A_0(Z_{\kappa(X)}, M_q, \omega^\vee_{X/k})
\end{tikzcd}$$

is commutative.

2.2.3. In particular, we have a well defined pairing

$$\cup : \widetilde{\mathrm{CH}}_0(Y_{\kappa(X)}, \omega^\vee_{X/k}) \otimes \widetilde{\mathrm{CH}}_0(Z_{\kappa(Y)}, \omega^\vee_{Y/k}) \to \widetilde{\mathrm{CH}}_0(Z_{\kappa(X)}, \omega^\vee_{X/k})$$

that can be taken for the composition law in the category of Milnor-Witt rational correspondences $\widetilde{\mathbf{RatCor}}(k)$ whose objects are the smooth proper schemes over k and morphisms are given by

$$\mathrm{Hom}_{\widetilde{\mathbf{RatCor}}(k)}(X,Y) = \bigoplus_i \widetilde{\mathrm{CH}}_0(Y_{\kappa(X_i)}, \omega^\vee_{X/k}),$$

where X_i are all irreducible (connected) components of X.

There is an obvious functor

$$\Xi : \widetilde{\mathbf{Cor}}(k) \to \widetilde{\mathbf{RatCor}}(k).$$

Theorem 2.2.4. *For an MW-cycle module M, there exists a well-defined contravariant functor*

$$\widetilde{\mathbf{RatCor}}(k) \to \mathscr{A}b, X \mapsto A_0(X,M), a \mapsto - \cup a.$$

More precisely, the functor $\widetilde{\mathbf{Cor}}(k) \to \widetilde{\mathbf{RatCor}}(k)$ factors through Ξ.

Proof. This follows from Proposition 2.2.1. □

Remark 2.2.5. Assuming one works with *oriented* (see A.2.2) smooth proper k-schemes, then there is also a contravariant functor given by $a \mapsto a \cup -$. We won't need this result.

2.2.6. If $\alpha : X \rightsquigarrow Y$ is a MW-rational correspondence between two smooth proper k-schemes, we have a natural pushforward morphism

$$\alpha_* : A_0(Y,M) \to A_0(X,M).$$

Remark 2.2.7. If α et β are two composable Milnor-Witt rational correspondences, then

$$(\alpha \circ \beta)_* = \alpha_* \circ \beta_*.$$

2.2.8. Let $f : Y \dashrightarrow X$ be a rational morphism of irreducible smooth k-schemes. It defines a rational point of $Y_{\kappa(X)}$ over $\kappa(X)$ and hence a morphism in $\mathrm{Hom}_{\widetilde{\mathbf{RatCor}}(k)}(X,Y)$ that we denote by $[f] : X \rightsquigarrow Y$. In fact, the rational correspondence $[f]$ is the image of the class of the (transposed of the) graph of f (as in [BCD+20, Chapter 2, §4.3]) under the natural map

$$\widetilde{\mathrm{CH}}_{d_X}(X \times Y, \omega_{X/k}) \to \widetilde{\mathrm{CH}}_0(Y_{\kappa(X)}, \omega_{X/k}).$$

Lemma 2.2.9. *Let κ/k be a finite type extension of fields. Let $f : X \dashrightarrow Y$ be a rational morphism of smooth proper κ-schemes and let $x \in X$ be a rational point such that $f(x)$ is defined. Denote by $[x] \in \widetilde{\mathrm{CH}}_0(X, \omega_{\kappa/k})$ the 0-cycle associated to x. Then*

$$[f]_*([x]) = [f(x)]$$

in $\widetilde{\mathrm{CH}}_0(Y, \omega_{\kappa/k})$.

Proof. Let $\Gamma \subset X \times Y$ be the graph of f. The preimage of $\{x\} \times \Gamma$ under the morphism $\Delta_X \otimes \mathrm{Id}_Y : X \times Y \to X \times X \times Y$ is the reduced scheme $\{x\} \times \{f(x)\}$. Hence

$$[f]_*([x]) = [x] \cup [f] = \pi_*(\Delta_X \otimes \mathrm{Id}_Y)^*([x] \times [\Gamma]) = \pi_*([x] \times [f(x)]) = [f(x)]$$

where $\pi : X \times Y \to Y$ is the projection. □

Corollary 2.2.10. *Let $f : X \dashrightarrow Y$ and $g : Y \dashrightarrow Z$ be composable rational morphisms of smooth proper schemes and let $h : X \dashrightarrow Z$ be the composition of f and g. Then $[g] \circ [f] = [h]$ in $\mathrm{Hom}_{\widetilde{\mathbf{RatCor}}(k)}(X, Z)$.*

Proof. Let y be the rational point of $Y_{\kappa(X)}$ corresponding to f. By assumption, the rational morphism $g_{\kappa(X)} : Y_{\kappa(X)} \dashrightarrow Z_{\kappa(X)}$ is defined at y. By Lemma 2.2.9 (with "$\kappa = \kappa(X)$","$X = Y_{\kappa(X)}$", "$Y = Z_{\kappa(X)}$ and "$f = g_{\kappa(X)}$") we see that the composition of correspondences f and g takes $[y]$ to $[g_{\kappa(X)}(y)] \in \widetilde{\mathrm{CH}}_0(Z_{\kappa(X)}, \omega^\vee_{X/k})$. Note that the latter class corresponds to h. □

Corollary 2.2.11. *For any two composable rational morphisms $f : X \dashrightarrow Y$ and $g : Y \dashrightarrow Z$ of smooth proper schemes, we have*

$$[g \circ f]_* = [g]_* \circ [f]_*.$$

Proof. This is a consequence of Corollary 2.2.10. □

Theorem 2.2.12. *The group $A_0(X, M)$ is a birational invariant of the smooth proper scheme X.*

In particular, the Chow-Witt group of zero-cycles $\widetilde{\mathrm{CH}}_0(X)$ is a birational invariant of the smooth proper scheme X.

Proof. This is an immediate consequence of Corollary 2.2.11. □

Example 2.2.13. According to [Fas20, §5], we know that $\widetilde{\mathrm{CH}}_0(\mathbf{P}^n_k) = \mathrm{GW}(k)$ if n is even, and $\widetilde{\mathrm{CH}}_0(\mathbf{P}^n_k) = \mathbf{Z}$ if n is odd.

In particular, we recover the computations of $\widetilde{\mathrm{CH}}_0(Q_n)$ where Q_n is an n-dimensional split quadric (see [HXZ20, Corollary 9.5]).

A Appendix

A.1 Cohomological Milnor-Witt cycle modules

Definition A.1.1. 1. If S is a scheme, call an S**-field** the spectrum of a field essentially of finite type over S, and a **morphism of** S**-fields** an S-morphism between the underlying schemes. The collection of S-fields together with morphisms of S-fields defines a category which we denote by $\mathcal{F}_S$. We say that a morphism of S-fields is **finite** (resp. **separable**) if the underlying field extension is finite (resp. separable).

In what follows, we will denote for example $f : \operatorname{Spec} F \to \operatorname{Spec} E$ a morphism of S-fields, and $\phi : E \to F$ the underlying field extension.

An S**-valuation** on an S-field $\operatorname{Spec} F$ is a discrete valuation v on F such that $\operatorname{Im}(\mathcal{O}(S) \to F) \subset \mathcal{O}_v$. We denote by $\kappa(v)$ the residue field, $\mathfrak{m}_v$ the valuation ideal and $N_v = \mathfrak{m}/\mathfrak{m}^2$.

2. Let S be a scheme and let R be a commutative ring with unit. An R**-linear cohomological Milnor-Witt cycle premodule** over S is a functor from $\mathcal{F}_S$ to the category of **Z**-graded R-modules

$$\begin{aligned} M : (\mathcal{F}_S)^{op} &\to \mathrm{Mod}_R^{\mathbf{Z}} \\ \operatorname{Spec} E &\mapsto M(E) \end{aligned} \tag{A.1.1.a}$$

for which we denote by $M_n(E)$ the n-the graded piece, together with the following functorialities and relations:

Functorialities:

(D1) For a morphism of S-fields $f : \operatorname{Spec} F \to \operatorname{Spec} E$ or (equivalently) $\phi : E \to F$, a map of degree 0

$$f^* = \phi_* = \mathrm{res}_{F/E} : M(E) \to M(F); \tag{A.1.1.b}$$

(D3) For an S-field $\operatorname{Spec} E$ and an element $x \in \mathrm{K}_m^{MW}(E)$, a map of degree m

$$\gamma_x : M(E) \to M(E) \tag{A.1.1.c}$$

making $M(E)$ a left module over the lax monoidal functor $\mathrm{K}_?^{MW}(E)$ (i.e. we have $\gamma_x \circ \gamma_y = \gamma_{x \cdot y}$ and $\gamma_1 = \mathrm{Id}$).

The axiom (D3) allows us to define, for every S-field $\operatorname{Spec} E$ and every 1-dimensional E-vector space $\mathcal{L}$, a graded R-module

$$M(E,\mathcal{L}) := M(E) \otimes_{R[E^\times]} R[\mathcal{L}^\times] \tag{A.1.1.d}$$

where $R[\mathcal{L}^\times]$ is the free R-module generated by the nonzero elements of $\mathcal{L}$, and the group algebra $R[E^\times]$ acts on $M(E)$ via $u \mapsto \langle u \rangle$ thanks to (D3).

(D2) For a finite morphism of S-fields $f : \operatorname{Spec} F \to \operatorname{Spec} E$ or $\phi : E \to F$, a map of degree 0

$$f_! = \phi^! = \operatorname{cores}_{F/E} : M(F, \omega_{F/E}) \to M(E); \tag{A.1.1.e}$$

(D4) For an S-field $\operatorname{Spec} E$ and an S-valuation v on E, a map of degree -1

$$\partial_v : M(E) \to M(\kappa(v), N_v^\vee). \tag{A.1.1.f}$$

Relations: We refer to [Fel20, Definition 3.1] for the list of relations.

A.1.2. Fix M a Milnor-Witt cycle premodule. If X is any scheme, let x, y be any points in X. We can define a map

$$\partial_y^x : M_q(\kappa(x), \omega_{\kappa(x)/k}) \to M_{q-1}(\kappa(y), \omega_{\kappa(y)/k})$$

thanks to (D2) and (D4).

Definition A.1.3. (see [Fel20, Definition 4.2])

A Milnor-Witt cycle module M over k is a Milnor-Witt cycle premodule M which satisfies the following conditions (FD) and (C).

(FD) FINITE SUPPORT OF DIVISORS. Let X be a normal scheme and ρ be an element of $M(\xi_X, X)$. Then $\partial_x(\rho) = 0$ for all but finitely many $x \in X^{(1)}$.

(C) CLOSEDNESS. Let X be integral and local of dimension 2. Then

$$0 = \sum_{x \in X^{(1)}} \partial_{x_0}^x \circ \partial_x^\xi : M(\kappa(\xi_X), \omega_{\kappa(\xi_X)/k}) \to M(\kappa(x_0), \omega_{\kappa(x_0)/k})$$

where ξ is the generic point and x_0 the closed point of X.

A.1.4. Let M be a Milnor-Witt cycle module over k. We can form a (cohomological) Rost-Schmid cycle complex $C_*(X, M, l)$ such that for any integer $p, q \in \mathbf{Z}$, and any line bundle l over X:

$$C_p(X, M_q, l) := \oplus_{X_{(p)}} M_{p+q}(\kappa(x), \omega_{\kappa(x)/k} \otimes l_{|x}). \tag{A.1.4.a}$$

We denote by $A_i(X, M_q, l)$ is the homology of $C_*(X, M_q, l)$ in degree i.

Remark A.1.5. Taking $M = \underline{\mathrm{K}}^{MW}$, we obtain

$$A_i(X, M_{-i}, l) = \widetilde{\mathrm{CH}}_i(X, l)$$

where the right-hand-side is known as the Chow-Witt group of X.

A.1.6. Fix M a Milnor-Witt cycle module and fix X a k-scheme with a dimensional pinning. We recall the basic maps that one can define on the cohomological Rost-Schmid complex.

A.1.7. PUSHFORWARD Let $f : Y \to X$ be a k-morphism of schemes. We have

$$f_* : C_p(Y, M_q, l) \to C_p(X, M_q, l)$$

as follows. If $x = f(y)$ and if $\kappa(y)$ is finite over $\kappa(x)$, then $(f_*)^y_x = \mathrm{cores}_{\kappa(y)/\kappa(x)}$. Otherwise, $(f_*)^y_x = 0$.

A.1.8. PULLBACK Let $f : Y \to X$ be an *essentially smooth* morphism of schemes of relative dimension s. Suppose Y connected. Define

$$f^! : C_p(X, M_q, l) \to C_{p+s}(Y, M_{q-s}, l \otimes \omega_f^\vee)$$

as follows. If $f(y) = x$, then $(f^!)^x_y = \mathrm{res}_{\kappa(y)/\kappa(x)}$. Otherwise, $(f^!)^x_y = 0$. If Y is not connected, take the sum over each connected component.

A.1.9. MULTIPLICATION WITH UNITS Let $a_1, \dots, a_n$ be global units in $\mathcal{O}_X^*$. Define

$$[a_1, \dots, a_n] : C_p(X, M_q, l) \to C_p(X, M_{q+n}, l)$$

as follows. Let x be in $X_{(p)}$ and $\rho \in \mathcal{M}(\kappa(x), *)$. We consider $[a_1(x), \dots, a_n(x)]$ as an element of $\underline{\mathrm{K}}^{MW}(\kappa(x))$. If $x = y$, then put $[a_1, \dots, a_n]^x_y(\rho) = [a_1(x), \dots, a_n(x)] \cdot \rho)$. Otherwise, put $[a_1, \dots, a_n]^x_y(\rho) = 0$.

A.1.10. MULTIPLICATION WITH η Define

$$\eta : C_p(X, M_q, l) \to C_p(X, M_{q-1}, l)$$

as follows. If $x = y$, then $\eta^x_y(\rho) = \gamma_\eta(\rho)$. Otherwise, $\eta^x_y(\rho) = 0$.

A.1.11. BOUNDARY MAPS Let X be a scheme of finite type over k, let $i : Z \to X$ be a closed immersion and let $j : U = X \setminus Z \to X$ be the inclusion of the open complement. We have a map

$$\partial = \partial^U_Z : C_p(U, M_q, *) \to C_{p-1}(Z, M_q, *).$$

which is called the boundary map for the closed immersion $i : Z \to X$.

A.1.12. A pairing $N \times M \to P$ between MW-cycle modules is given by maps

$$M_p(E, l) \otimes N_q(E, l') \to P_{p+q}(E, l \otimes l')$$

which are compatible with the data (D1),..., (D4) (see [Fel20, Definition 3.21] for more details).

A.1.13. PRODUCT If $M \times N \to P$ is a pairing of Milnor-Witt cycle modules, then there is a product map

$$C_p(X, M_q, l) \times C_r(Y, N_s, l') \to C_{p+r}(X \times Y, P_{q+s}, l \otimes l')$$

where X, Y are smooth schemes over k (see also [Fel20, §11]).

Remark A.1.14. The previous basic maps commute with the differentials of the Rost-Schmid complex and thus induce morphisms on the homology.

A.2 Oriented schemes

A.2.1. The notion of oriented real vector bundles was extended to the algebraic setting by Barges-Morel in [BM00]. We introduce a new category of oriented schemes. We refer to [DDØ22, Appendix §6.1] for similar results.

Definition A.2.2. Let X/S be a scheme. An orientation of X is an isomorphism $\sigma : \omega_{X/S} \to l_X^{\otimes 2}$, where l_X is an invertible sheaf over X.

An oriented S-scheme $(X, \sigma_X : \omega_{X/S} \to l_X^{\otimes 2})$ is the data of a scheme X/S and an orientation $\sigma_X : \omega_{X/S} \to l_X^{\otimes 2}$.

A morphism of oriented schemes $(Y, \sigma_Y : \omega_{Y/S} \to l_Y^{\otimes 2}) \to (X, \sigma_X : \omega_{X/S} \to l_X^{\otimes 2})$ is the data of an S-morphism $f : Y \to X$ along with an isomorphism of invertible sheaves $l_Y^{\otimes 2} \simeq f^{-1} l_X^{\otimes 2} \otimes \omega_f$ which makes the following diagram

$$\begin{tikzcd}
\omega_{Y/S} \arrow[r, "\simeq"] \arrow[d, "\sigma_Y"] & f^{-1}\omega_{X/S} \otimes \omega_f \arrow[d, "\sigma_X \otimes \mathrm{Id}_{\omega_f}"] \\
l_Y^{\otimes 2} \arrow[r, "\simeq"] & f^{-1} l_X^{\otimes 2} \otimes \omega_f
\end{tikzcd}$$

commutative.

Denote by **orSchm** the category of oriented schemes (along with morphisms of oriented schemes).

Remark A.2.3. Let $(X, \sigma_X : \omega_{X/S} \to l_X^{\otimes 2})$ be an oriented scheme. By abuse of notation, we omit the orientation and simply write X.

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